

一个涉及多重可変上限函数及部分和的 半离散 Mulholland 型不等式

曾志红¹, 杨必成²

(1. 广东第二师范学院 学报编辑部, 广州 510303; 2. 广东第二师范学院 数学学院, 广州 510303)

摘要: 应用权函数方法、Euler-Maclaurin 求和公式、Abel 求部分和公式及微分中值定理, 给出一个涉及多重可変上限函数及部分和的半离散 Mulholland 型不等式, 并讨论新不等式
中最佳常数因子联系多参数的等价性.

关键词: 权系数; 参数; 半离散 Mulholland 型不等式; Euler-Maclaurin 求和公式; 部分和;
Abel 求部分和公式

中图分类号: O178 **文献标志码:** A **文章编号:** 1671-5489(2026)03-0511-10

A Half-Discrete Mulholland-Type Inequality Involving Multiple Variable Upper Limit Function and Partial Sums

ZENG Zhihong¹, YANG Bicheng²

(1. Editorial Department of Journal, Guangdong University of Education, Guangzhou 510303, China;
2. School of Mathematics, Guangdong University of Education, Guangzhou 510303, China)

Abstract: By using the weight function method, the Euler-Maclaurin summation formula, Abel's summation by parts formula and the differential mean value theorem, we gave a new half-discrete Mulholland-type inequality involving multiple upper limit function and partial sums, and discussed the equivalence of the best constant factor with multiple parameters in the new inequality.

Keywords: weight coefficient; parameter; half-discrete Mulholland-type inequality; Euler-Maclaurin summation formula; partial sums; Abel's summation by parts formula

0 引言

设 $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, 则有如下具有最佳常数因子

$\frac{\pi}{\sin(\pi/p)}$ 的 Hardy-Hilbert 不等式^[1]:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \quad (1)$$

具有相同最佳常数因子的 Mulholland 不等式^[1]:

收稿日期: 2025-07-07.

第一作者简介: 曾志红(1974—), 女, 汉族, 编审, 从事解析不等式的研究, E-mail: zhzh181@126.com. **通信作者简介:** 杨必成(1946—), 男, 汉族, 教授, 从事解析不等式的研究, E-mail: bcyang@gdei.edu.cn.

基金项目: 国家自然科学基金(批准号: 61772140).

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_mb_n}{\ln(mn)}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{1/q},\tag{2}$$

及如下半离散 Mulholland 型不等式(取 $\lambda=1, v(n)=\ln n, r=q, s=p$)^[2]:

$$\int_b^{\infty}\sum_{n=2}^{\infty}\frac{f(x)b_n}{u(x)+\ln n}<\frac{\pi}{\sin(\pi/p)}\left(\int_b^{\infty}\frac{f^p(x)}{(u'(x))^{p-1}}\mathrm{d}x\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{1/q}.\tag{3}$$

Krnić等^[3]给出式(1)的如下推广形式:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m+n)^{\lambda}}<\mathrm{B}(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{1/p}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{1/q},\tag{4}$$

这里 $\lambda_i\in(0,2](i=1,2), \lambda_1+\lambda_2=\lambda\in(0,4], \mathrm{B}(\lambda_1,\lambda_2)$ 是最佳常数, $\mathrm{B}(u,v):=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\mathrm{d}t$ ($u,v>0$) 为 Beta 函数. 当 $p=q=2, \lambda_1=\lambda_2=\lambda/2$ 时, 式(3)变为文献[4]的一个不等式. 文献[5-12]给出了不等式(1)~(3)的一些推广及应用.

目前, 关于不等式(1)~(3)的研究已有很多结果^[13-25]. 如洪勇等^[13]讨论了式(1)的一般形式联系最佳常数因子及多参量的等价条件. 沿用文献[3]的方法, Adiyasuren 等^[18]给出了如下关于式(3)的涉及部分和的拓展性推广: 设 $\lambda_i\in(0,1]\cap(0,\lambda)(\lambda\in(0,2], i=1,2), \lambda_1+\lambda_2=\lambda$, 则有

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m+n)^{\lambda}}<\lambda_1\lambda_2\mathrm{B}(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q\right)^{1/q},\tag{5}$$

这里 $\lambda_1\lambda_2\mathrm{B}(\lambda_1,\lambda_2)$ 为最佳值, $a_m,b_n\geqslant 0, A_m=\sum_{i=1}^ma_i, B_n=\sum_{k=1}nb_k(m,n\in N=\{1,2,\cdots\})$, 满足条件

$A_m=o(e^{tm}), B_n=o(e^{tn})(t>0, m,n\rightarrow\infty)$, 及 $0<\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p<\infty, 0<\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q<\infty$. 最近,

Yang 等^[19]应用 Euler-Maclaurin 求和公式, 通过引入参量给出了式(5)的联系核 $\frac{1}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}}$

$(\alpha,\beta\in(0,1], \xi,\eta\in[0,\frac{1}{5}])$ 较精确的推广. 王爱珍等^[20]给出了联系核为 $\frac{1}{(x+n^{\beta})^{\lambda+m}}$ 涉及 m -阶导函数

及部分和的半离散 Hilbert 型不等式.

本文应用权函数的方法、Euler-Maclaurin 求和公式及 Abel 求部分和公式, 给出一个涉及多重可变上限函数及部分和的半离散 Mulholland 型不等式, 它是式(3)当 $u(x)=x$ 时的推广, 并讨论了新不等式中最佳常数因子联系多参数的等价性.

1 引 理

本文假设条件如下:

$$\begin{aligned} &(\mathrm{H}_1) \quad p>1(q>1), \frac{1}{p}+\frac{1}{q}=1, \beta\in(0,1], \lambda>0, \lambda_1\in(0,\lambda), \lambda_2\in\left(0,\frac{2}{\beta}-1\right]\cap(0,\lambda), \hat{\lambda}_1:=\\ &\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}, f(x):=F_0(x) \text{ 在 } \mathbb{R}_+=(0,\infty) \text{ 上除有限点外非负连续}; \end{aligned}$$

$$(\mathrm{H}_2) \quad \text{对于 } m\in N, \text{ 定义多重可变上限函数 } F_i(x):=\int_0^x F_{i-1}(t)\mathrm{d}t(\geqslant 0)(x\geqslant 0), \text{ 即}$$

$$F_i(x)=\int_0^x\left(\int_0^{t_{i-1}}\cdots\int_0^{t_1}F_0(t_0)\mathrm{d}t_0\cdots\mathrm{d}t_{i-2}\right)\mathrm{d}t_{i-1}(i=1,2,\cdots,m),$$

满足条件 $F_i(x)=o(e^{tx})(t>0, x\rightarrow\infty)(i=1,2,\cdots,m)$, 对于 $m\in\mathbb{N}=\{0,1,2,\cdots\}$, 有 $0<$

$$\int_0^{\infty}t^{p(1-m-\hat{\lambda}_1)-1}F_m^p(t)\mathrm{d}t<\infty;$$

$$(\mathrm{H}_3) \quad \text{设 } b_n\geqslant 0, B_n:=\sum_{k=2}^nb_k(n\in N_2=\{2,3,\cdots\}), \text{ 满足条件 } B_n=o(e^{t\ln^{\beta}n})(t>0, n\rightarrow\infty) \text{ 及 } 0<$$

$$\sum_{n=2}^{\infty}\frac{\ln^{-q\hat{\lambda}_2-1}nB_n^q}{n}<\infty.$$

引理 1^[5] 1) 若 $k \in \mathbb{N}$, $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [k, \infty), g^{(i)}(\infty) = 0 (i = 0, 1), P_1(t) := t - [t] - 1/2 (t \in \mathbb{R} = (-\infty, \infty))$ 为一阶 Bernoulli 函数, 则有

$$-\frac{1}{8}g(k) < \int_k^\infty P_1(t)g(t)dt < 0; \tag{6}$$

2) 若 $f(t) \in C^1[k, \infty)$, 则有如下 Euler-Maclaurin 求和公式:

$$\sum_{i=k}^n f(i) = \int_k^n f(t)dt + \frac{1}{2}(f(k) + f(n)) + \int_k^n P_1(t)f'(t)dt (n \geq k). \tag{7}$$

引理 2 设 $s > 0, s_2 \in (0, 2/\beta] \cap (0, s)$, 权系数定义为

$$\bar{\omega}_s(s_2, x) := x^{s-s_2} \sum_{n=2}^\infty \frac{\beta \ln^{\beta s_2 - 1} n}{(x + \ln^\beta n)^s n} (x > 0). \tag{8}$$

则有如下双向不等式:

$$0 < B(s_2, s - s_2)(1 - O_s(1/x^{s_2})) < \bar{\omega}_s(s_2, x) < B(s_2, s - s_2)(x > 0). \tag{9}$$

这里 $O_s\left(\frac{1}{x^{s_2}}\right) := \frac{1}{k_s(s_2)} \int_0^{(\ln^{\beta 2})/x} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

证明: 固定 $x > 0$, 定义正函数 $h(t) := \frac{\ln^{\beta s_2 - 1} t}{(x + \ln^\beta t)^s t} (t > 1)$. 下面分两步证明.

1) 当 $s_2 \in (0, 1/\beta] \cap (0, s), \beta \in (0, 1]$ 时, 求导可得

$$h'(t) = -\frac{(1 - \beta s_2) \ln^{\beta s_2 - 2} t}{(x + \ln^\beta t)^s t^2} - \frac{\beta s \ln^{\beta(s_2+1) - 2} t}{(x + \ln^\beta t)^{s+1} (t - \eta)^2} - \frac{\ln^{\beta s_2 - 1} t}{(x + \ln^\beta t)^s t^2} < 0.$$

由级数的递减性质, 有

$$\int_2^\infty h(t)dt < \sum_{n=2}^\infty h(n) < \int_1^\infty h(t)dt. \tag{10}$$

2) 当 $s_2 \in (1/\beta, 2/\beta] \cap (0, s), h(\infty) = 0$ 时, 由式(7) (取 $k = 2, n \rightarrow \infty$) 有

$$\begin{aligned} \sum_{n=2}^\infty h(n) &= \int_2^\infty h(t)dt + \frac{1}{2}h(2) + \int_2^\infty P_1(t)h'(t)dt = \int_1^\infty h(t)dt - h_0(s_2), \\ h_0(s_2) &:= \int_1^2 h(t)dt - \frac{1}{2}h(2) - \int_2^\infty P_1(t)h'(t)dt, \end{aligned}$$

从而可得 $-\frac{1}{2}h(2) = -\frac{\ln^{\beta s_2 - 1} 2}{4(x + \ln^{\beta 2})^s}$. 做变换 $u = \ln^\beta t$, 由分部积分法有

$$\begin{aligned} \int_1^2 h(t)dt &= \frac{1}{\beta} \int_0^{\ln^{\beta 2}} \frac{u^{s_2-1}}{(x+u)^s} du = \frac{1}{\beta} \int_0^{\ln^{\beta 2}} \frac{du^{s_2}}{s_2(x+u)^s} = \\ &= \frac{u^{s_2}}{\beta s_2(x+u)^s} \Big|_0^{\ln^{\beta 2}} - \frac{1}{\beta} \int_0^{\ln^{\beta 2}} \frac{u^{s_2}}{s_2} d\frac{1}{(x+u)^s} = \frac{\ln^{\beta s_2} 2}{\beta s_2(x + \ln^{\beta 2})^s} + \\ &= \frac{s}{\beta s_2(s_2+1)} \int_0^{\ln^{\beta 2}} \frac{du^{s_2+1}}{(x+u)^{s+1}} > \frac{\ln^{\beta s_2} 2}{\beta s_2(x + \ln^{\beta 2})^s} + \frac{s}{\beta s_2(s_2+1)} \frac{\ln^{\beta(s_2+1)} 2}{(x + \ln^{\beta 2})^{s+1}}. \end{aligned}$$

若 $(-1)^{i-1} f_j^{(i-1)}(t) > 0 (i, j = 1, 2)$, 则显然有 $(-1)^{i-1} (f_1(t)f_2(t))^{(i-1)} > 0 (i = 1, 2)$. 基于上述结果及 $\frac{\ln t}{t^2} > 0$ 和 $\left(\frac{\ln t}{t^2}\right)' = \frac{1-2\ln t}{t^3} < 0 (t > 2)$, 由式(6)有

$$\begin{aligned} h'(t) &= \frac{(\beta s_2 - 1) \ln^{\beta s_2 - 2} t}{(x + \ln^\beta t)^s t^2} - \frac{\beta s \ln^{\beta s_2 - (3-\beta)} t}{(9x + \ln^\beta t)^{s+1}} \frac{\ln t}{t^2} - \frac{\ln^{\beta s_2 - 2} t}{(x + \ln^\beta t)^s} \frac{\ln t}{t^2}, \\ &\quad - \int_2^\infty P_1(t) \frac{(\beta s_2 - 1) \ln^{\beta s_2 - 2} t}{(x + \ln^\beta t)^s t^2} dt > 0 (1 < \beta s_2 \leq 2), \\ \int_2^\infty P_1(t) &\left[\frac{\beta s \ln^{\beta s_2 - (3-\beta)} t}{(x + \ln^\beta t)^{s+1}} \frac{\ln t}{t^2} + \frac{\ln^{\beta s_2 - 2} t}{(x + \ln^\beta t)^s} \frac{\ln t}{t^2} \right] dt > \\ &- \frac{1}{8} \left[\frac{\beta s \ln^{\beta s_2 - (3-\beta)} 2}{(x + \ln^{\beta 2})^{s+1}} \frac{\ln 2}{2^2} + \frac{\ln^{\beta s_2 - 2} 2}{(x + \ln^{\beta 2})^s} \frac{\ln 2}{2^2} \right] (\beta s_2 - (3 - \beta) \leq 2 - 2 = 0), \end{aligned}$$

$$-\int_2^\infty P_1(t)h'(t)dt > -\frac{\beta s \ln^{\beta s_2-(2-\beta)} 2}{32(x+\ln^\beta 2)^{s+1}} - \frac{\ln^{\beta s_2-1} 2}{32(x+\ln^\beta 2)^s}.$$

从而对于 $s_2 \in (1/\beta, 2/\beta] \cap (0, s)$, $\beta \in (0, 1]$, 可得

$$\begin{aligned} h_0(s_2) &> \frac{\ln^{\beta s_2} 2}{\beta s_2 (x+\ln^\beta 2)^s} + \frac{s}{\beta s_2 (s_2+1)} \frac{\ln^{\beta(s_2+1)} 2}{(x+\ln^\beta 2)^{s+1}} - \\ &\quad \frac{\ln^{\beta s_2-1} 2}{4(x+\ln^\beta 2)^s} - \frac{\beta s \ln^{\beta(s_2+1)-1} 2}{32(x+\ln^\beta 2)^{s+1}} - \frac{\beta \ln^{\beta s_2-1} 2}{32(x+\ln^\beta 2)^s} = \\ &\quad \frac{\ln^{\beta s_2-1} 2}{(x+\ln^\beta 2)^s} \left(\frac{\ln 2}{\beta s_2} - \frac{1}{4} - \frac{1}{32} \right) + \frac{s \beta \ln^{\beta(s_2+1)-1} 2}{(x+\ln^\beta 2)^{s+1}} \left[\frac{\ln 2}{\beta s_2 (\beta s_2 + \beta)} - \frac{1}{32} \right] \geq \\ &\quad \frac{\ln^{\beta s_2-1} 2}{(x+\ln^\beta 2)^s} \left(\frac{\ln 2}{2} - \frac{1}{4} - \frac{1}{32} \right) + \frac{s \beta \ln^{\beta(s_2+1)-1} 2}{(x+\ln^\beta 2)^{s+1}} \left(\frac{\ln 2}{6} - \frac{1}{32} \right) > \\ &\quad \frac{\ln^{\beta s_2-1} 2}{(x+\ln^\beta 2)^s} 0.065 + \frac{s \beta \ln^{\beta(s_2+1)-1} 2}{(x+\ln^\beta 2)^{s+1}} 0.084 > 0. \end{aligned}$$

此外, 还有

$$\sum_{n=2}^\infty h(n) = \int_2^\infty h(t)dt + \frac{1}{2}h(2) + \int_2^\infty P_1(t)h'(t)dt = \int_2^\infty h(t)dt + h_1(s_2),$$

这里 $h_1(s_2) := \frac{1}{2}h(2) + \int_2^\infty P_1(t)h'(t)dt$. 对于 $s_2 \in (1/\beta, 2/\beta] \cap (0, s)$, 由式(6)有

$$\begin{aligned} &\int_2^\infty P_1(t) \frac{(\beta s_2 - 1) \ln^{\beta s_2-2} t}{(x+\ln^\beta t)^s t^2} dt > -\frac{(\beta s_2 - 1) \ln^{\beta s_2-2} 2}{32(x+\ln^\beta 2)^s}, \\ &-\int_2^\infty P_1(t) \left[\frac{\beta s \ln^{\beta s_2-(3-\beta)} t}{(x+\ln^\beta t)^{s+1}} \frac{\ln t}{t^2} + \frac{\ln^{\beta s_2-2} t}{(x+\ln^\beta t)^s} \frac{\ln t}{t^2} \right] dt > 0, \\ &\int_2^\infty P_1(t)h'(t)dt > -\frac{(\beta s_2 - 1) \ln^{\beta s_2-2} 2}{32(x+\ln^\beta 2)^s}, \\ h_1(s_2) &> \frac{\ln^{\beta s_2-1} 2}{4(x+\ln^\beta 2)^s} - \frac{(\beta s_2 - 1) \ln^{\beta s_2-2} 2}{32(x+\ln^\beta 2)^s} = \frac{\ln^{\beta s_2-2} 2}{4(x+\ln^\beta 2)^s} \left(\ln 2 - \frac{\beta s_2 - 1}{8} \right) \geq \\ &\quad \frac{\ln^{\beta s_2-2} 2}{4(x+\ln^\beta 2)^s} \left(\ln 2 - \frac{1}{8} \right) > \frac{\ln^{\beta s_2-2} 2}{4(x+\ln^\beta 2)^s} 0.568 > 0, \end{aligned}$$

故式(10)成立.

对于 $s_2 \in [0, 2/\beta] \cap (0, s)$, 由式(10), 做变换 $u = (\ln^\beta t)/x$, 可得

$$\begin{aligned} \bar{\omega}_s(s_2, x) &= \beta x^{s-s_2} \sum_{n=2}^\infty h(n) < \beta x^{s-s_2} \int_1^\infty h(t)dt = \int_0^\infty \frac{u^{s_2-1} du}{(1+u)^s} = B(s_2, s-s_2), \\ \bar{\omega}_s(s_2, x) &> \beta x^{s-s_2} \int_2^\infty h(t)dt = \int_{(\ln^\beta 2)/x}^\infty \frac{u^{s_2-1} du}{(1+u)^s} = B(s_2, s-s_2) \left(1 - O_s \left(\frac{1}{x^{s_2}} \right) \right) > 0, \end{aligned}$$

这里 $O_s \left(\frac{1}{x^{s_2}} \right) = \frac{1}{k_s(s_2)} \int_0^{(\ln^\beta 2)/x} \frac{u^{s_2-1}}{(1+u)^s} du$, 满足如下不等式:

$$0 < \int_0^{(\ln^\beta 2)/x} \frac{u^{s_2-1}}{(1+u)^s} du \leq \int_0^{(\ln^\beta 2)/x} u^{s_2-1} du = \frac{1}{s_2} \left(\frac{\ln^\beta 2}{x} \right)^{s_2}.$$

故式(9)成立. 证毕.

引理 3 假设条件 $(H_1) \sim (H_3)$ 成立, 则有如下半离散不等式:

$$\begin{aligned} I_{m+1} &:= \sum_{n=2}^\infty \int_0^\infty \frac{\ln^{\beta-1} n}{(x+\ln^\beta n)^{\lambda+m+1}} \frac{1}{n} F_m(x) B_n dx < \\ &\quad \left(\frac{1}{\beta} B(\lambda_2+1, \lambda-\lambda_2+m) \right)^{1/p} (B(\lambda_1+m, \lambda-\lambda_1+1))^{1/q} \times \\ &\quad \left[\int_0^\infty x^{p(1-m-\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \left(\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1} n}{n} B_n^q \right)^{1/q}. \end{aligned} \tag{11}$$

证明: 做变换 $y = \frac{x}{\ln^\beta n}$, 对于 $s_1 \in (0, s) (s > 0)$, 可得下列权函数表达式:

$$\omega_s(s_1, n) := \ln^{\beta(s-s_1)} n \int_0^\infty \frac{x^{s_1-1}}{(x + \ln^\beta n)^s} dx = \int_0^\infty \frac{y^{s_1-1}}{(y+1)^s} dy = B(s_1, s-s_1) \quad (n \in N_2). \tag{12}$$

由 Hölder 不等式^[26], 可得

$$\begin{aligned} I_{m+1} &= \sum_{n=2}^\infty \int_0^\infty \frac{1}{(x + \ln^\beta n)^{\lambda+m+1}} \left[\frac{\ln^{\lceil \beta(\lambda_2+1)-1 \rceil / p} n}{n^{1/p} x^{(\lambda_1+m-1)/q}} F_m(x) \right] \left[\frac{x^{(\lambda_1+m-1)/q}}{\ln^{\lceil \beta(\lambda_2+1)-1 \rceil / p} n} \frac{\ln^{\beta-1} n}{n^{1/q}} B_n \right] dx \leqslant \\ &\quad \left[\int_0^\infty \sum_{n=2}^\infty \frac{1}{(x + \ln^\beta n)^{\lambda+m+1}} \frac{\ln^{\beta(\lambda_2+1)-1} n}{x^{(\lambda_1+m-1)(p-1)} n} F_m^p(x) dx \right]^{1/p} \times \\ &\quad \left[\sum_{n=2}^\infty \int_0^\infty \frac{1}{(x + \ln^\beta n)^{\lambda+m+1}} \frac{x^{\lambda_1+m-1}}{\ln^{\lceil \beta(\lambda_2+1)-1 \rceil (q-1)} n} dx \frac{\ln^{q(\beta-1)} n}{n} B_n^q \right]^{1/q} = \\ &\quad \left[\frac{1}{\beta} \int_0^\infty \bar{\omega}_{\lambda+m+1}(\lambda_2+1, x) x^{p(1-m-\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \times \\ &\quad \left[\sum_{n=2}^\infty \omega_{\lambda+m+1}(\lambda_1+m, n) \frac{\ln^{-q\beta\lambda_2-1} n}{n} B_n^q \right]^{1/q}. \end{aligned}$$

由式(9)及式(12), 取 $s=\lambda+m+1>m+1$, $s_1=\lambda_1+m \in (m, \lambda+m)$, $s_2=\lambda_2+1 \in (1, 2/\beta] \cap (1, \lambda+1)$, 则有 $\lambda>0$, $\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, 2/\beta-1] \cap (0, \lambda)$ 及式(11). 证毕.

引理 4 假设条件 $(H_1) \sim (H_3)$ 成立, 对于 $t>0$, $m \in \mathbb{N}$, 则有

$$\int_0^\infty e^{-tx} f(x) dx = t^m \int_0^\infty e^{-tx} F_m(x) dx. \tag{13}$$

证明: 对于 $m=0$, 因为 $F_0(x)=f(x)$, 故式(13)成立. 由已知条件, 对于 $m \in N$, $i=1, 2, \cdots, m$, 有 $F_i(0^+)=0$ 及 $e^{-tx} F_i(x)|_0^\infty=0$. 由分部积分法, 得

$$\int_0^\infty e^{-tx} F_{i-1}(x) dx = \int_0^\infty e^{-tx} dF_i(x) = e^{-tx} F_i(x)|_0^\infty - \int_0^\infty F_i(x) de^{-tx} = t \int_0^\infty e^{-tx} F_i(x) dx. \tag{14}$$

在式(14)中代入 $i=1, 2, \cdots, m(m \in N)$, 化简可得式(13). 证毕.

引理 5 假设条件 $(H_1) \sim (H_3)$ 成立, 对于 $t>0$, 则有如下离散不等式:

$$\sum_{n=2}^\infty e^{-t\ln^\beta n} b_n \leqslant t\beta \sum_{n=2}^\infty e^{-t\ln^\beta n} \frac{\ln^{\beta-1} n}{n} B_n. \tag{15}$$

证明: 因为 $B_n e^{-t\ln^\beta(n+1)} = B_n e^{-t\ln^\beta n} = o(1) (n \rightarrow \infty)$, 故由 Abel 求部分和公式, 有

$$\begin{aligned} \sum_{n=2}^\infty e^{-t\ln^\beta n} b_n &= \lim_{N \rightarrow \infty} \left\{ B_N e^{-t\ln^\beta(N+1)} + \sum_{n=2}^N B_n [e^{-t\ln^\beta n} - e^{-t\ln^\beta(n+1)}] \right\} = \\ &= \sum_{n=2}^\infty B_n [e^{-t\ln^\beta n} - e^{-t\ln^\beta(n+1)}]. \end{aligned}$$

设 $f(x)=e^{-t\ln^\beta x}$, $x \in (1, \infty)$, 则可得 $f'(x)=-t\beta \frac{\ln^{\beta-1} x}{x} e^{-t\ln^\beta x} = -t\beta h(x)$, 其中对于 $\beta \in (0, 1]$, 函数

$h(x) := \frac{\ln^{\beta-1} x}{x} e^{-t\ln^\beta x}$ 在 $(1, \infty)$ 内严格递减且连续可导. 由微分中值定理, 有

$$\begin{aligned} \sum_{n=2}^\infty e^{-t\ln^\beta(n-\eta)} b_n &= - \sum_{n=2}^\infty B_n (f(n+1) - f(n)) = - \sum_{n=2}^\infty B_n f'(n+\theta_n) = t\beta \sum_{n=2}^\infty h(n+\theta_n) B_n \leqslant \\ &= t\beta \sum_{n=2}^\infty h(n) B_n = t\beta \sum_{n=2}^\infty e^{-t\ln^\beta n} \frac{\ln^{\beta-1} n}{n} B_n \quad (\theta_n \in (0, 1)), \end{aligned}$$

故式(15)成立. 证毕.

引理 6 假设条件 $(H_1) \sim (H_3)$ 成立, 则有如下不等式:

$$I := \int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{(x + \ln^\beta n)^\lambda} dx \leqslant \beta \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty \frac{\ln^{\beta-1} n}{(x + \ln^\beta n)^{\lambda+m+1} n} B_n F_m(x) dx. \tag{16}$$

证明: 利用 Gamma 函数表示式^[27] $\frac{1}{(x + \ln^\beta n)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(x+\ln^\beta n)t} dt$, 由 Lebesgue 逐项积分定理^[28] 及式(13), (15), 有

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty b_n f(x) \int_0^\infty t^{\lambda-1} e^{-(x+\ln^\beta n)t} dt dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(\int_0^\infty e^{-xt} f(x) dx \right) \left(\sum_{n=2}^\infty e^{-t \ln^\beta n} b_n \right) dt \leqslant \\ &\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(t^m \int_0^\infty e^{-xt} F_m(x) dx \right) \left(t \beta \sum_{n=2}^\infty e^{-t \ln^\beta n} \frac{\ln^{\beta-1} n}{n} B_n \right) dt = \\ &\beta \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty \frac{\ln^{\beta-1} n}{n} B_n F_m(x) \int_0^\infty t^{(\lambda+m+1)-1} e^{-(x+\ln^\beta n)t} dt dx = \\ &\beta \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty \frac{\ln^{\beta-1} n}{(x+\ln^\beta n)^{\lambda+m+1}} B_n F_m(x) dx. \end{aligned}$$

故式(16)成立. 证毕.

引理 7 对于 $b \in (-1, 1)$, $n \in N_2$, 有常数 C , 使得

$$\sum_{k=2}^n \frac{\ln^b k}{k} = \frac{\ln^{b+1} n}{b+1} + C + O\left(\frac{1}{n} \ln^b n\right) \quad (n \rightarrow \infty). \tag{17}$$

证明: 设函数 $G(t) := \frac{1}{t} \ln^b t \ (t \geqslant 2)$, 求导得 $G'(t) = \frac{b}{t^2} \ln^{b-1} t - \frac{1}{t^2} \ln^b t = b g_1(t) - g_2(t)$, 其中 $g_1(t) := \frac{1}{t^2} \ln^{b-1} t$, $g_2(t) := \frac{1}{t^2} \ln^b t \ (t \geqslant 2)$. 则有 $(-1)^i g_1^{(i)}(t) > 0 \ (t \geqslant 2, i=0, 1)$. 因为

$$b < 1 < 2 \ln 2 \leqslant 2 \ln t \ (t \geqslant 2), \quad g_2'(t) = \frac{b-2 \ln t}{t^3} \ln^{b-1} t < 0 \ (t \geqslant 2),$$

故有 $(-1)^i g_2^{(i)}(t) > 0 \ (i=0, 1)$. 由式(6), 有

$$\int_2^n P_1(t) g_j(t) dt = \frac{\epsilon_j}{8} g_j(t) \Big|_2^n \quad (0 < \epsilon_j < 1, j=0, 1).$$

由式(7), 有

$$\begin{aligned} \sum_{k=2}^n G(k) &= \int_2^n G(t) dt + \frac{1}{2} (G(2) + G(n)) + \int_2^n P_1(t) G'(t) dt = \\ &\int_2^n G(t) dt + \frac{1}{2} (G(2) + G(n)) + b \int_2^n P_1(t) g_1(t) dt - \int_2^n P_1(t) g_2(t) dt. \end{aligned}$$

再代入上述相关结果, 有式(17). 这里 $C := -\frac{1}{b+1} \ln^{b+1} 2 + \left(\frac{1}{4} + \frac{\epsilon_2}{32}\right) \ln^b 2 - \frac{\epsilon_1 b}{32} \ln^{b-1} 2, O\left(\frac{1}{n} \ln^b n\right) := \frac{\ln^b n}{2n} + \frac{\epsilon_1 b}{8n^2} \ln^{b-1} n - \frac{\epsilon_2}{8n^2} \ln^b n \ (n \rightarrow \infty)$. 证毕.

2 主要结果

定理 1 假设条件 $(H_1) \sim (H_3)$ 成立, 则有如下涉及多重可变上限函数及部分和的半离散 Mulholland 型不等式:

$$\begin{aligned} I &= \int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{(x+\ln^\beta n)^\lambda} dx < \beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} (B(\lambda_2+1, \lambda-\lambda_2+m))^{1/p} \times \\ &(B(\lambda_1+m, \lambda-\lambda_1+1))^{1/q} \left[\int_0^\infty x^{\beta(1-m-\lambda_1)-1} F_m^\beta(x) dx \right]^{1/p} \left(\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1} n}{n} B_n^q \right)^{1/q}. \end{aligned} \tag{18}$$

特别地, 当 $\lambda_1 + \lambda_2 = \lambda$ 时, 由假设有

$$\begin{aligned} 0 &< \int_0^\infty x^{\beta(1-m-\lambda_1)-1} F_m^\beta(x) dx < \infty, \quad 0 < \sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1} n}{n} B_n^q < \infty, \\ \int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{(\lfloor x+\ln^\beta n \rfloor)^\lambda} dx &< \beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1+m, \lambda_2+1) \times \\ &\left[\int_0^\infty x^{\beta(1-m-\lambda_1)-1} F_m^\beta(x) dx \right]^{1/p} \left(\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1} n}{n} B_n^q \right)^{1/q}, \end{aligned} \tag{19}$$

这里常数因子 $\beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1+m, \lambda_2+1)$ 为最佳值.

证明: 由式(16)及式(11)有式(18). $\forall 0 < \epsilon < \min\{p\lambda_1, q\lambda_2\}$, 设

$$\tilde{F}_0(x) = \tilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 - \epsilon/p - 1}, & x \geq 1, \end{cases} \quad \tilde{b}_n := \frac{1}{n} \ln^{\beta(\lambda_2 - \epsilon/q) - 1} n, \quad n \in N_2,$$

则对于 $m \in \mathbb{N}$, 可得

$$\begin{aligned} \tilde{F}_m(x) &:= \int_0^x \left(\int_0^{t_{m-1}} \cdots \int_0^{t_1} \tilde{f}(t_0) dt_0 \cdots dt_{m-2} \right) dt_{m-1} = \\ &\begin{cases} 0, & 0 < x < 1, \\ \frac{1}{\prod_{i=0}^{m-1} \left(\lambda_1 + i - \frac{\epsilon}{p} \right)} (x^{\lambda_1 + m - 1 - \epsilon/p} - p_{m-1}(x)), & x \geq 1, \end{cases} \end{aligned} \quad (20)$$

这里 $p_{m-1}(x)$ 为正系数的 $(m-1)$ 次多项式, 满足 $p_{m-1}^{(i-1)}(1) = (x^{\lambda_1 + m - 1 - \epsilon/p})^{(i-1)}|_{x=1} (i=1, 2, \cdots, m)$. 记

$p_{-1}(x) := 0, \prod_{i=0}^{-1} (a+i) := 1 (a > 0)$. 式(20)对 $m \in \mathbb{N}$ 成立, $\tilde{F}_i(x) = o(e^{tx}) (t > 0, x \rightarrow \infty)$

$(i=1, 2, \cdots, m)$. 对于 $-1 < b = \beta(\lambda_2 - \epsilon/q) - 1 < 1$, 由式(17)有

$$\begin{aligned} 0 < \tilde{B}_n &:= \sum_{k=2}^n \tilde{b}_k = \sum_{k=2}^n \frac{1}{k} \ln^{\beta(\lambda_2 - \epsilon/q) - 1} k = \frac{1}{\beta(\lambda_2 - \epsilon/q)} \ln^{\beta(\lambda_2 - \epsilon/q)} n + C + o_1 \left(\frac{1}{n} \ln^{\beta(\lambda_2 - \epsilon/q) - 1} n \right) \leq \\ &\frac{\ln^{\beta(\lambda_2 - \epsilon/q)} n}{\beta(\lambda_2 - \epsilon/q)} \left[1 + |C_1| \ln^{\beta(\lambda_2 - \epsilon/q)} n + \left| o_1 \left(\frac{1}{n \ln n} \right) \right| \right], \end{aligned}$$

满足条件 $\tilde{B}_n = o(e^{t \ln^{\beta} n}) (t > 0, n \rightarrow \infty)$.

对于可微函数 $a(x) = o(1) (x \rightarrow \infty)$, 由 L'Hôpital 法则, 有

$$\lim_{x \rightarrow \infty} \frac{(1+a(x))^q - 1}{a(x)} \left(\frac{0}{0} \right) = \lim_{x \rightarrow \infty} \frac{q(1+a(x))^{q-1} a'(x)}{a'(x)} = \lim_{x \rightarrow \infty} q(1+a(x))^{q-1} = q,$$

即有 $(1+a(x))^q = 1 + o(a(x)) (x \rightarrow \infty)$, 从而有

$$\left(1 + |C_1| \ln^{-\beta(\lambda_2 - \epsilon/q)} n + \left| o \left(\frac{1}{n \ln n} \right) \right| \right)^q = 1 + o \left(|C_1| \ln^{-\beta(\lambda_2 - \epsilon/q)} n + \left| o_1 \left(\frac{1}{n \ln n} \right) \right| \right) (n \in N_2, n \rightarrow \infty).$$

于是有

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2-1}}{n} n \tilde{B}_n^q &\leq \left[\frac{1}{\beta(\lambda_2 - \epsilon/q)} \right]^q \sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\epsilon-1} n \left(1 + |C_1| \ln^{-\beta(\lambda_2 - \epsilon/q)} n + \left| o_1 \left(\frac{1}{n \ln n} \right) \right| \right)^q = \\ &\left[\frac{1}{\beta(\lambda_2 - \epsilon/q)} \right]^q \sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\epsilon-1} n \left[1 + o \left(|C_1| \ln^{-\beta(\lambda_2 - \epsilon/q)} n + \left| o_1 \left(\frac{1}{n \ln n} \right) \right| \right) \right] = \\ &\left[\frac{1}{\beta(\lambda_2 - \epsilon/q)} \right]^q \left[\sum_{n=2}^{\infty} \frac{1}{n} \ln^{-\beta\epsilon-1} n + \sum_{n=2}^{\infty} o \left(|C_1| \frac{1}{n} \ln^{-\beta(\lambda_2 - \epsilon/q)} n + \left| o_1 \left(\frac{1}{n \ln n} \right) \right| \right) \right] < \\ &\left[\frac{1}{\beta(\lambda_2 - \epsilon/q)} \right]^q \left(\int_2^{\infty} \ln^{-\beta\epsilon-1} x d \ln x + O_1(1) \right) = \left[\frac{1}{\beta(\lambda_2 - \epsilon/q)} \right]^q \left(\frac{1}{\beta\epsilon} \ln^{-\beta\epsilon} 2 + O_1(1) \right). \end{aligned}$$

若有正常数 $M \leq \beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\lambda_1+m, \lambda_2+1)$, 使其取代式(19)中的常数因子 $\beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} \times$

$B(\lambda_1+m, \lambda_2+1)$ 后仍成立, 则有

$$\tilde{I} := \int_0^{\infty} \sum_{n=2}^{\infty} \frac{\tilde{b}_n \tilde{f}(x)}{(x + \ln^{\beta} n)^{\lambda}} dx < M \left[\int_0^{\infty} x^{p(1-m-\lambda_1)-1} \tilde{F}_m^p(x) dx \right]^{1/p} \left(\sum_{n=2}^{\infty} \frac{\ln^{-q\beta\lambda_2-1}}{n} n \tilde{B}_n^q \right)^{1/q}. \quad (21)$$

由式(21)及上述结果, 有

$$\tilde{I} < \frac{M \left[\int_1^{\infty} x^{p(1-m-\lambda_1)-1} x^{\beta(\lambda_1+m-1)-\epsilon} dx \right]^{1/p}}{\beta \left(\lambda_2 - \frac{\epsilon}{q} \right) \prod_{i=0}^{m-1} \left(\lambda_1 + i - \frac{\epsilon}{p} \right)} \left(\frac{1}{\beta\epsilon} \ln^{-\beta\epsilon} 2 + O_1(1) \right)^{1/q} =$$

$$\frac{M}{\beta\left(\lambda_2-\frac{\epsilon}{q}\right)\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\epsilon}{p}\right)}\left(\int_1^\infty x^{-\epsilon-1}\mathrm{d}x\right)^{1/p}\left(\frac{1}{\beta\epsilon}\ln^{-\beta\epsilon}2+O_1(1)\right)^{1/q}=\\ \frac{M}{\epsilon\beta\left(\lambda_2-\frac{\epsilon}{q}\right)\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\epsilon}{p}\right)}\left(\frac{1}{\beta}\ln^{-\beta\epsilon}2+\epsilon O_1(1)\right)^{1/q}.$$

由式(8)及式(9), 设 $s=\lambda\in(0,\infty)$, $s_2=\tilde{\lambda}_2:=\lambda_2-\epsilon/q\in(0,2/\beta)\cap(0,\lambda)$, 则有

$$\tilde{I}=\int_1^\infty\left[x^{\lambda_1+\epsilon/q}\sum_{n=2}^\infty\frac{\ln^{\beta(\lambda_2-\epsilon/q)-1}n}{(x+\ln^\beta n)^{\lambda}n}\right]x^{-\epsilon-1}\mathrm{d}x=\frac{1}{\beta}\int_1^\infty\bar{\omega}_\lambda(\tilde{\lambda}_2,x)x^{-\epsilon-1}\mathrm{d}x>\\ \frac{1}{\beta}k_\lambda(\tilde{\lambda}_2)\int_1^\infty\left(1-O_\lambda\left(\frac{1}{x^{\tilde{\lambda}_2}}\right)\right)x^{-\epsilon-1}\mathrm{d}x=\frac{1}{\beta}\mathrm{B}\left(\lambda_1+\frac{\epsilon}{q},\lambda_2-\frac{\epsilon}{q}\right)\left(\frac{1}{\epsilon}-O(1)\right).$$

基于上述结果, 有

$$\frac{1}{\beta}\mathrm{B}\left(\lambda_1+\frac{\epsilon}{q},\lambda_2-\frac{\epsilon}{q}\right)-\epsilon O(1)<\epsilon\tilde{I}<\frac{M(\beta^{-1}\ln^{-\beta\epsilon}2+\epsilon O_1(1))^{1/q}}{\beta\left(\lambda_2-\frac{\epsilon}{q}\right)\prod_{i=0}^{m-1}\left(\lambda^1+i-\frac{\epsilon}{p}\right)}.$$

(22)

当 $\epsilon\rightarrow 0^+$ 时, 由 Beta 函数的连续性质, 式(22)变为

$$\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\mathrm{B}(\lambda_1+m,\lambda_2+1)=\beta^{1/q}\lambda_2\mathrm{B}(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)\leqslant M.$$

故 $M=\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\mathrm{B}(\lambda_1+m,\lambda_2+1)$ 为式(19)的最佳常数因子. 证毕.

定理 2 假设条件 $(\mathrm{H}_1)\sim(\mathrm{H}_3)$ 成立, 若式(18)的常数因子

$$\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(\mathrm{B}(\lambda_2+1,\lambda-\lambda_2+m))^{1/p}(\mathrm{B}(\lambda_1+m,\lambda-\lambda_1+1))^{1/q}$$

是最佳值, 则当 $\lambda-\lambda_1-\lambda_2\leqslant q(2/\beta-1-\lambda_2)$ 时, 有 $\lambda_1+\lambda_2=\lambda$.

证明: 因为 $\hat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2=\frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, $0<\lambda_1,\lambda_2<\lambda$, 故有 $\hat{\lambda}_1+\hat{\lambda}_2=\lambda$, $0<\hat{\lambda}_1,\hat{\lambda}_2<\lambda$,

$\mathrm{B}(\hat{\lambda}_1+m,\hat{\lambda}_2+1)\in\mathbb{R}_+$. 当 $\lambda-\lambda_1-\lambda_2\leqslant q\left(\frac{2}{\beta}-1-\lambda_2\right)$ 时, 有 $\hat{\lambda}_2\leqslant\frac{2}{\beta}-1$. 将 $\lambda_i=\hat{\lambda}_i$ ($i=1,2$) 代入式(19), 有如下不等式:

$$\int_0^\infty\sum_{n=2}^\infty\frac{b_nf(x)}{(x+\ln^\beta n)^\lambda}\mathrm{d}x<\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\mathrm{B}(\hat{\lambda}_1+m,\hat{\lambda}_2+1)\times\\ \left[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1}F_m^p(x)\mathrm{d}x\right]^{1/p}\left(\sum_{n=2}^\infty\frac{\ln^{-q\beta\hat{\lambda}_2-1}n}{n}nB_n^q\right)^{1/q}.$$

(23)

由 Hölder 不等式, 有

$$\mathrm{B}(\hat{\lambda}_1+m,\hat{\lambda}_2+1)=k_{\lambda+m+1}\left(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}+1\right)=\int_0^\infty\frac{1}{(1+u)^{\lambda+m+1}}u^{\lambda_2/p+(\lambda-\lambda_1)/q}\mathrm{d}u=\\ \int_0^\infty\frac{1}{(1+u)^{\lambda+m+1}}(u^{\lambda_2/p})(u^{(\lambda-\lambda_1)/q})\mathrm{d}u\leqslant\\ \left[\int_0^\infty\frac{1}{(1+u)^{\lambda+m+1}}u^{\lambda_2}\mathrm{d}u\right]^{1/p}\left[\int_0^\infty\frac{1}{(1+u)^{\lambda+m+1}}u^{\lambda-\lambda_1}\mathrm{d}u\right]^{1/q}=\\ \left[\int_0^\infty\frac{1}{(1+u)^{\lambda+m+1}}u^{\lambda_2}\mathrm{d}u\right]^{1/p}\left[\int_0^\infty\frac{1}{(1+v)^{\lambda+m+1}}v^{(\lambda_1+m)-1}\mathrm{d}v\right]^{1/q}=\\ (\mathrm{B}(\lambda_2+1,\lambda-\lambda_2+m))^{1/p}(\mathrm{B}(\lambda_1+m,\lambda-\lambda_1+1))^{1/q}.$$

(24)

由假设条件知, 常数因子 $\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(\mathrm{B}(\lambda_2+1,\lambda-\lambda_2+m))^{1/p}(\mathrm{B}(\lambda_1+m,\lambda-\lambda_1+1))^{1/q}$ 为式(18)的最佳值, 故比较式(18)及式(23)的常数因子, 有如下不等式:

$$\beta^{1/q}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(\mathrm{B}(\lambda_2+1,\lambda-\lambda_2+m))^{1/p}(\mathrm{B}(\lambda_1+m,\lambda-\lambda_1+1))^{1/q}\leqslant$$

$$\beta^{1/q} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) (\in \mathbb{R}_+),$$

即有

$$B(\hat{\lambda}_1+m, \hat{\lambda}_2+1) \geq (B(\lambda_2+1, \lambda-\lambda_2+m))^{1/p} (B(\lambda_1+m, \lambda-\lambda_1+1))^{1/q}.$$

故式(24)取等号. 式(24)取等号的等价条件是有不全为0的常数A和B, 使得 $Au^{\lambda_2} = Bu^{\lambda-\lambda_1}$ a. e. 于 $\mathbb{R}_+^{[26]}$. 不妨设 $A \neq 0$, 则有 $u^{\lambda_2+\lambda_1-\lambda} = B/A$ a. e. 于 $\mathbb{R}_+$, $\lambda_2+\lambda_1-\lambda=0$. 从而有 $\lambda_1+\lambda_2=\lambda$. 证毕.

注1 在式(19)中, 取 $m=0$, 则有如下具有最佳常数因子 $\beta^{1/q}\lambda_2 B(\lambda_1, \lambda_2)$ 的不等式:

$$\int_0^\infty \sum_{n=2}^\infty \frac{b_n f(x)}{(x+\ln^\beta n)^\lambda} dx < \beta^{1/q} \lambda_2 B(\lambda_1, \lambda_2) \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{1/p} \left(\sum_{n=2}^\infty \frac{\ln^{-q\beta\lambda_2-1} n B_n^q}{n} \right)^{1/q}. \quad (25)$$

故式(19)(式(18))是式(25)的推广.

参 考 文 献

- [1] HARDY G H, LITTLEWOOD J E, PÓLYA G. Inequalities [M]. Cambridge: Cambridge University Press, 1934: 234-235.
- [2] CHEN Q, YANG B C. Half-Discrete Hardy-Hilbert's Inequality with Two Internal Variables [J/OL]. Journal of Inequalities and Applications, (2013-11-07)[2025-06-26]. <https://doi.org/10.1186/1029-242x-2013-485>.
- [3] KRNIĆ M, PEČARIĆ J. Extension of Hilbert's Inequality [J]. Journal of Mathematical Analysis and Applications, 2006, 324(1): 150-160.
- [4] 杨必成. 关于 Hilbert 重级数定理的一个推广 [J]. 南京大学学报(数学半年刊), 2001, 18(1): 145-152. (YANG B C. On a Generalization of Hilbert Double Series Theorem [J]. Journal of Nanjing University (Mathematical Biquarterly), 2001, 18(1): 145-152.)
- [5] 杨必成. 算子范数与 Hilbert 型不等式 [M]. 北京: 科学出版社, 2009: 1-371. (YANG B C. The Norm of Operator and Hilbert-Type Inequalities [M]. Beijing: Science Press, 2009: 1-371.)
- [6] KRNIĆ M, PEČARIĆ J. General Hilbert's and Hardy's Inequalities [J]. Mathematical Inequalities & Applications, 2005, 8(1): 29-51.
- [7] PERIĆ I, VUKOVIĆ P. Multiple Hilbert's Type Inequalities with a Homogeneous Kernel [J]. Banach Journal of Mathematical Analysis, 2011, 5(2): 33-43.
- [8] 杨必成. 一个加强型逆向的 Hardy-Littlewood-Polya 不等式 [J]. 五邑大学学报(自然科学版), 2025, 39(3): 57-64. (YANG B C. A Strengthened Version of a Reverse Hardy-Littlewood-Polya's Inequality [J]. Journal of Wuyi University (Natural Science Edition), 2025, 39(3): 57-64.)
- [9] AZAR L E. The Connection between Hilbert and Hardy Inequalities [J/OL]. Journal of Inequalities and Applications, (2013-11-07)[2025-06-26]. <https://doi.org/10.1186/1029-242x-2013-452>.
- [10] ADIYASUREN V, BATBOLD T, KRINIĆ M. Hilbert-Type Inequalities Involving Differential Operators, the Best Constants, and Applications [J]. Mathematical Inequalities & Applications, 2015, 18(1): 111-124.
- [11] 谢子填, 曾峥. 一个 -4μ 齐次新的半离散 Hilbert 不等式 [J]. 湛江师范学院学报, 2011, 32(6): 13-19. (XIE Z T, ZENG Z. A New Half-Discrete Hilbert's Inequality with the Homogeneous Kernel of Degree -4μ [J]. Journal of Zhanjiang Normal College, 2011, 32(6): 13-19.)
- [12] 钟五一. 一个混合类的 Hilbert 型不等式 [J]. 广东第二师范学院学报, 2011, 31(5): 18-22. (ZHONG W Y. A Mixed Hilbert-Type Inequality and Its Equivalent Forms [J]. Journal of Guangdong University of Education, 2011, 31(5): 18-22.)
- [13] 洪勇, 温雅敏. 齐次核的 Hilbert 型级数不等式取最佳常数因子的充要条件 [J]. 数学年刊 A 辑(中文版), 2016, 37A(3): 329-336. (HONG Y, WEN Y M. A Necessary and Sufficient Condition of That Hilbert Type Series Inequality with Homogeneous Kernel Has the Best Constant Factor [J]. Chinese Annals of Mathematics, 2016, 37A(3): 329-336.)
- [14] 洪勇. 具有齐次核的 Hilbert 型积分不等式的构造特征及应用 [J]. 吉林大学学报(理学版), 2017, 55(2): 189-194. (HONG Y. Structural Characteristics and Applications of Hilbert's Type Integral Inequality with

- Homogeneous Kernel [J]. Journal of Jilin University (Science Edition), 2017, 55(2): 189-194.)
- [15] HE B, HONG Y, LI Z. Conditions for the Validity of a Class of Optimal Hilbert Type Multiple Integral Inequalities with Nonhomogeneous Kernels [J/OL]. Journal of Inequalities and Applications, (2021-04-02) [2025-06-26]. <https://doi.org/10.1186/s13660-021-02593-z>.
- [16] CHEN Q, HE B, HONG Y, et al. Equivalent Parameter Conditions for the Validity of Half-Discrete Hilbert-Type Multiple Integral Inequality with Generalized Homogeneous Kernel [J/OL]. Journal of Function Spaces, (2020-08-11)[2025-06-26]. <https://doi.org/10.1155/2020/7414861>.
- [17] HE B, HONG Y, CHEN Q. The Equivalent Parameter Conditions for Constructing Multiple Integral Half-Discrete Hilbert-Type Inequalities with a Class of Nonhomogeneous Kernels and Their Applications [J]. Open Mathematics, 2021, 19: 400-411.
- [18] ADIYASUREN V, BATBOLD T, AZAR L E. A New Discrete Hilbert-Type Inequality Involving Partial Sums [J/OL]. Journal of Inequalities and Applications, (2019-05-08)[2025-06-26]. <https://doi.org/10.1186/s13660-019-2087-6>.
- [19] YANG B C, WU S H. An Improved Version of the Parameterized Hardy-Hilbert Inequality Involving Two Partial Sums [J]. Mathematics, 2025, 13(8): 1331-1-1331-14.
- [20] 王爱珍, 杨必成. 一个新的涉及高阶导函数与部分和的半离散 Hilbert 型不等式 [J]. 吉林大学学报(理学版), 2023, 61(6): 1296-1304. (WANG A Z, YANG B C. A New Half-Discrete Hilbert-Type Inequality Involving Higher-Order Derivative Function and Partial Sum [J]. Journal of Jilin University (Science Edition), 2023, 61(6): 1296-1304.)
- [21] HUANG X Y, YANG B C. On a More Accurate Half-Discrete Mulholland-Type Inequality Involving One Multiple Upper Limit Function [J/OL]. Journal of Function Spaces, (2021-12-07) [2025-06-20]. <https://doi.org/10.1155/2021/6970158>.
- [22] PENG L, ABD RAHIM R, YANG B C. A New Reverse Half-Discrete Mulholland-Type Inequality with a Nonhomogeneous Kernel [J/OL]. Journal of Inequalities and Applications, (2023-09-13)[2025-06-28]. <https://doi.org/10.1186/s13660-023-03025-w>.
- [23] HUANG X Y, LUO R C, YANG B C, et al. A New Reverse Mulholland's Inequality with One Partial Sum in the Kernel [J/OL]. Journal of Inequalities and Applications, (2024-01-24) [2025-06-28]. <https://doi.org/10.1186/s13660-024-03080-x>.
- [24] PENG L, YANG B C. A New Extended Mulholland's Inequality Involving One Partial Sum [J]. Open Mathematics, 2024, 22(1): 150-160.
- [25] HONG Y, ZHANG L J, XIAO H S. Condition for the Construction of a Hilbert-Type Integral Inequality Involving Upper Limit Functions [J]. Symmetry, 2024, 16(12): 1682-1-1682-10.
- [26] 匡继昌. 常用不等式 [M]. 济南: 山东科技出版社, 2004: 1-857. (KUANG J C. Applied Inequalities [M]. Jinan: Shangdong Science and Technology Press, 2004: 1-857.)
- [27] 王竹溪, 郭敦仁. 特殊函数概论 [M]. 北京: 科学出版社, 1965: 1-300. (WANG Z X, GUO D R. Introduction to Special Function [M]. Beijing: Science Press, 1965: 1-300.)
- [28] 匡继昌. 实分析与泛函分析: 续集, 第2卷 [M]. 北京: 高等教育出版社, 2015: 1-336. (KUANG J C. Real and Functional Analysis: Continuation, Vol. 2 [M]. Beijing: Higher Education Press, 2015: 1-336.)

(责任编辑: 赵立芹)