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双分散介质流体在无界区域上的空间性质

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摘要：首先，利用微分不等式技术给出双向流动介质在牛顿冷却边界条件下关于温度的 L^4 范数和解的先验估计；其次，利用解的先验估计并设置适当的能量函数，证明半无限管道中解随空间变量代数式衰减.

关键词：双向流动介质；空间性质；能量分析
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Spatial Properties of Bidisperse Media Flow in Unbounded Domain

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Abstract: Firstly, by using differential inequality techniques, we gave a prior estimate of the L^4 norm and solution of temperature for bidirectional flow media under the Newtonian cooling boundary conditions. Secondly, by using a prior estimate of the solution and setting an appropriate energy function, we proved that the solutions decayed algebraically with spatial variable in a semi-infinite pipe.

Keywords: bidispersive flow; spatial property; energy analysis

0 引 言

双分散介质是一种多孔体，具有通常的大孔，但在固体骨架中也存在裂缝或裂纹，从而产生微孔[1]. 利用 Nield 等[2-4]的一般理论，Falsaperla 等[5]导出了单温度双扩散多孔介质中的热对流方程. 在不失一般性的情况下，Franchi 等[1]将双分散(或双重孔隙)介质扩散模型简化为以下形式：

$$b_1\boldsymbol{u}+\nabla p+a(\boldsymbol{u}-\boldsymbol{v})-\boldsymbol{g}T=0,\qquad \nabla\cdot\boldsymbol{u}=0, \tag{1}$$

$$b_2\boldsymbol{v}+\nabla q-a(\boldsymbol{u}-\boldsymbol{v})-\boldsymbol{g}T=0,\qquad \nabla\cdot\boldsymbol{v}=0, \tag{2}$$

$$T_{,t}-\Delta T+c(\boldsymbol{u}+\boldsymbol{v})\cdot\nabla T=0, \tag{3}$$

其中：$\boldsymbol{u}=(u_1,u_2,u_3)$和$\boldsymbol{v}=(v_1,v_2,v_3)$表示流体的速度；$T$ 表示温度；p,q 表示压强；a 表示相容系数；$\boldsymbol{g}=(g_1,g_2,g_3)$表示重力函数，不失一般性，假设$|\boldsymbol{g}|\leqslant 1$；$c$ 表示流体的热膨胀系数；b_1,b_2 表示饱和流体的动态黏度. 在双向流动中包含温度非常重要，因为热效应可能会在固体骨架中引发裂纹，进而导致微孔. 因此，本文在模型中考虑了温度的影响. 多孔介质中的双向流动模型(1)-(3)在流体力学中应用广泛. 例如，油藏开采[6]、化学工程[7-9]、土地排水和确保雨水径流不污染方面[10-11]、滑坡及其对人

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类生命的灾难性影响[12]等. Castro 等[13]证明了模型(1)-(3)强解的存在性.

由于在模型建立和简化的过程中，不可避免地出现一些误差，因此考察这些误差是否会对模型的解产生巨大影响非常必要. Franchi 等[1]研究了三维有界域 Ω 上解的结构稳定性，其中解在区域的边界上满足

$$T=h(x_1,x_2,x_3,t),\quad \boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{v}\cdot\boldsymbol{n}=0,\quad \text{在 } \partial\Omega\times\{t>0\} \text{ 上},$$

或者牛顿冷却边界条件

$$\frac{\partial T}{\partial \boldsymbol{n}}+\kappa T=F(\boldsymbol{x},t),\qquad \text{在 } \partial\Omega\times\{t>0\} \text{ 上},$$

这里 $\boldsymbol{n}$ 和 $\dfrac{\partial}{\partial \boldsymbol{n}}$ 分别表示 $\partial\Omega$ 上的单位外法向量和单位外法向导数，h 和 F 是已知函数，κ 表示牛顿冷却系数. 本文研究双分散介质扩散模型(1)-(3)通过一个半无穷柱体时的空间性质. 目前，关于各种类型流体模型和弹性模型解的空间性质研究已得到广泛关注[14-20]，但对双分散介质扩散模型解的空间性质研究尚未见文献报道. 基于此，本文研究双分散介质流体在无界区域上的空间性质. 令 R 表示一个底面位于 x_1ox_2 坐标平面上的半无穷柱体，即

$$R=\{\boldsymbol{x}\,|\,x_3>0,(x_1,x_2)\in D\},$$

其中 D 是坐标平面 x_1ox_2 上的一个光滑有界凸区域. 假设模型(1)～(3)的解满足以下边界条件：

$$T=H,\quad \boldsymbol{u}=\boldsymbol{f},\quad \boldsymbol{v}=\boldsymbol{h},\qquad \text{在 } D\times\{t>0\} \text{ 上},\tag{4}$$

$$\frac{\partial T}{\partial \boldsymbol{n}}+\kappa T=F(\boldsymbol{x},t),\quad \boldsymbol{u}=\boldsymbol{v}=0,\qquad \text{在 } \partial D\times\{x_3>0\}\times\{t>0\} \text{ 上},\tag{5}$$

$$T(x_1,x_2,x_3,0)=0,\qquad \text{在 } R \text{ 上},\tag{6}$$

$$|\nabla T|,|T|=O(1),\quad u_i,v_i,|q|,|p|=o(x_3^{-1}),\quad x_3\to\infty,\tag{7}$$

其中：$\boldsymbol{f}=(f_1,f_2,f_3)$，$\boldsymbol{h}=(h_1,h_2,h_3)$ 和 H 是大于零的已知函数，∂D 表示 D 的边界.

与文献[1]相比，本文不仅要考虑时间变量而且还要考虑空间变量. 因此，本文采用空间加权的方法. 由于在柱体侧面上考虑了牛顿冷却边界条件和空间变量的影响，因此本文不采用文献[21-22]中的方法获得温度 T 的上确界. 本文的第一个创新点是通过设置适当的辅助函数，推导出温度 T 的上界. 为控制 T 在 R 上的 L^4 范数，传统方法通常利用以下 Sobolev 不等式：

$$\int_R|\phi|^4\mathrm{d}x\leqslant\Lambda\left(\int_R\phi_{,j}\phi_{,j}\mathrm{d}x\right)^2,\qquad \Lambda>0,$$

其中 ϕ 是一个 Dirichlet 可积函数，$\phi|_{\partial D}=0$，$\phi\to\infty(x_3\to\infty)$[19,23]. 显然，该不等式在本文中不再成立. 本文的第二个创新点是推导一个关于 T 在 R 上的 L^4 范数的先验估计.

符号约定：D_z 表示 R 在 $x_3=z$ 处的横截面，即 $D_z=\{\boldsymbol{x}\,|\,x_3=z\geqslant0,(x_1,x_2)\in D\}$；令 R_z 表示 R 的子集，即 $R_z=\{\boldsymbol{x}\,|\,x_3\geqslant z\geqslant0,(x_1,x_2)\in D\}$；用逗号表示求导；重复英文字母表示从 1 至 3 求和，重复希腊字母表示从 1 至 2 求和，例如 $u_{i,j}u_{i,j}=\sum\limits_{i,j=1}^{3}\left(\dfrac{\partial u_i}{\partial x_j}\right)^2$，$u_{\alpha,\beta}u_{\alpha,\beta}=\sum\limits_{\alpha,\beta=1}^{2}\left(\dfrac{\partial u_\alpha}{\partial x_\beta}\right)^2$.

1　微分不等式

为推导模型(1)-(7)解的先验界，给出以下引理.

引理 1[24]　若 ϕ 是 D 上的 Dirichlet 可积函数，且 $\int_D\phi\mathrm{d}A=0$，则存在函数 $\varphi=(\varphi_1,\varphi_2)$，满足

$$\varphi_{\alpha,\alpha}=\phi,\quad \text{在 } D \text{ 上};\qquad \varphi_\alpha=0,\quad \text{在 } \partial D \text{ 上}.$$

并存在仅依赖于 D 的大于零的常数 k_1，满足

$$\int_D\varphi_{\alpha,\beta}\varphi_{\alpha,\beta}\mathrm{d}A\leqslant k_1\int_D(\varphi_{\alpha,\alpha})^2\mathrm{d}A.$$

引理 2[24]　如果 $\phi|_{\partial D}=0$，则

$$\lambda\int_D\phi^2\mathrm{d}A\leqslant\int_D\phi_{,\alpha}\phi_{,\alpha}\mathrm{d}A,$$

其中 λ 是问题 $v_{,\alpha\alpha}+\lambda v=0$ 在 D 上及 $v=0$ 在 ∂D 上的第一特征值.

如果 $\phi|_{\partial D}=0$ 不成立，则可利用以下引理.

引理 3[25] 如果 ϕ 是二维区域 D 上的有界光滑函数，则存在依赖于 ∂D 的常数 $k_2>0$，满足

$$\int_D|\phi|^2\mathrm{d}A\leqslant k_2\left(\int_{\partial D}\phi^2\mathrm{d}l+\int_D\phi_{,\alpha}\phi_{,\alpha}\mathrm{d}A\right).$$

引理 4 如果 ϕ 是一个 Dirichlet 可积函数，且当 $x_3\to\infty$ 时，$\phi\to\infty$，则

$$\left(\int_{R_z}|\phi|^4\mathrm{d}\boldsymbol{x}\right)^{1/2}\leqslant k_3\left(\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}+\int_{R_z}\phi_{,i}\phi_{,i}\mathrm{d}\boldsymbol{x}\right),\qquad k_3>0.$$

证明：利用 Hölder 不等式，可得

$$\int_{R_z}|\phi|^4\mathrm{d}\boldsymbol{x}\leqslant\int_z^\infty\left(\int_{D_\xi}\phi^6\ \mathrm{d}A\right)^{1/2}\left(\int_{D_\xi}\phi^2\ \mathrm{d}A\right)^{1/2}\mathrm{d}\xi.\tag{8}$$

令 P 是 D_ξ 内的任一点，P_1，P_2 和 Q_1，Q_2 是 ∂D_ξ 分别与通过点 P 的直线 $x_2=C_1$(常数)和 $x_1=C_2$(常数)的交点. 由于

$$\phi^3(P)=\phi(P_1)^3+3\int_{P_1}^P\phi^2\phi_{,1}\mathrm{d}x_1=-3\int_P^{P_2}\phi^2\phi_{,1}\mathrm{d}x_1+\phi(P_2)^3,$$

所以

$$\phi^3(P)\leqslant\frac{1}{2}(|\phi(P_2)|^3+|\phi(P_1)|^3)+\frac{3}{2}\int_{P_1}^{P_2}|\phi_{,\alpha}|\phi^2\mathrm{d}x_1.\tag{9}$$

同理，有

$$\phi^3(P)\leqslant\frac{1}{2}(|\phi(Q_2)|^3+|\phi(Q_1)|^3)+\frac{3}{2}\int_{Q_1}^{Q_2}\phi^2|\phi_{,\alpha}|\mathrm{d}x_2.\tag{10}$$

结合式(9)，(10)，有

$$\int_{D_\xi}\phi^6\mathrm{d}A\leqslant\left(\frac{1}{2}\int_{\partial D_\xi}\phi^3\mathrm{d}l+\frac{3}{2}\int_{D_\xi}\phi^2|\phi_{,\alpha}|\mathrm{d}A\right)^2.\tag{11}$$

令 $p_0=\min\limits_{\partial D_\xi}(x_\alpha n_\alpha)$，$d_0=\max\limits_{\overline{D_\xi}}(x_\alpha x_\alpha)$，则有

$$p_0\int_{\partial D_\xi}\phi^3\mathrm{d}l\leqslant\int_{\partial D_\xi}x_\alpha n_\alpha\phi^3\mathrm{d}l\leqslant2\int_{D_\xi}\phi^3\mathrm{d}A+3d_0\int_{D_\xi}\phi^2|\phi_{,\alpha}|\mathrm{d}A.$$

从而由式(11)可得

$$\int_{D_\xi}\phi^6\mathrm{d}A\leqslant\left[\frac{1}{p_0}\int_{D_\xi}\phi^3\mathrm{d}A+\frac{3(1+d_0)}{2p_0}\int_{D_\xi}\phi^2|\phi_{,\alpha}|\mathrm{d}A\right]^2.\tag{12}$$

结合式(8)，(12)，有

$$\int_{R_z}\phi^4\mathrm{d}\boldsymbol{x}\leqslant\left[\frac{1}{p_0}\int_{R_z}\phi^3\mathrm{d}\boldsymbol{x}+\frac{3(1+d_0)}{2p_0}\int_{R_z}\phi^2|\phi_{,\alpha}|\mathrm{d}\boldsymbol{x}\right]\max_{z\leqslant\xi\leqslant\infty}\left(\int_{D_\xi}\phi^2\ \mathrm{d}A\right)^{1/2}.\tag{13}$$

注意到

$$\int_{D_\xi}\phi^2\mathrm{d}A=2\int_\xi^\infty\int_{D_\tau}\phi\phi_{,3}\mathrm{d}A\mathrm{d}\tau\leqslant2\left(\int_{R_\xi}\phi^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{R_\xi}\phi_{,3}^2\mathrm{d}\boldsymbol{x}\right)^{1/2},\tag{14}$$

把式(14)代入式(13)并利用 Hölder 不等式，可得

$$\begin{aligned}\int_{R_z}|\phi|^4\mathrm{d}\boldsymbol{x}\leqslant{}&\sqrt{2}\left[\frac{1}{p_0}\left(\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{R_z}\phi^4\mathrm{d}\boldsymbol{x}\right)^{1/2}+\frac{3(1+d_0)}{2p_0}\left(\int_{R_z}\phi^4\mathrm{d}x\right)^{1/2}\left(\int_{R_z}\phi_{,\alpha}\phi_{,\alpha}\mathrm{d}\boldsymbol{x}\right)^{1/2}\right]\times\\&\left(\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}\right)^{1/4}\left(\int_{R_z}\phi_{,3}^2\mathrm{d}\boldsymbol{x}\right)^{1/4}\leqslant\\&\sqrt{2}\left(\int_{R_z}\phi^4\mathrm{d}\boldsymbol{x}\right)^{1/2}\left[\frac{1}{p_0}\left(\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}\right)^{3/4}\left(\int_{R_\xi}\phi_{,3}^2\mathrm{d}\boldsymbol{x}\right)^{1/4}+\frac{3(1+d_0)}{2p_0}\left(\int_{R_z}\phi_{,i}\phi_{,i}\mathrm{d}\boldsymbol{x}\right)^{3/4}\left(\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}\right)^{1/4}\right]\leqslant\\&\sqrt{2}\left(\int_{R_z}\phi^4\mathrm{d}\boldsymbol{x}\right)^{1/2}\left[\frac{3}{4p_0}\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}+\frac{1}{4p_0}\int_{R_\xi}\phi_{,3}^2\mathrm{d}\boldsymbol{x}+\right.\\&\left.\frac{9(1+d_0)}{8p_0}\int_{R_z}\phi_{,i}\phi_{,i}\mathrm{d}\boldsymbol{x}+\frac{3(1+d_0)}{8p_0}\int_{R_z}\phi^2\mathrm{d}\boldsymbol{x}\right].\end{aligned}\tag{15}$$

在式(15)中取 $k_3=\sqrt{2}\max\left\{\frac{3}{4p_0},\frac{9(1+d_0)}{8p_0}\right\}$ 即可完成证明.

2　先验界

设 $\widetilde{H}^{[p]}(x_1,x_2,x_3,t)$ 是问题

$$\Delta\widetilde{H}^{[p]}(x_1,x_2,x_3,t)=0,\qquad \text{在 } R\times\{t>0\} \text{ 上},\tag{16}$$

$$\widetilde{H}^{[p]}(x_1,x_2,t)=H^{2p-1}(x_1,x_2,t),\qquad \text{在 } D\times\{t>0\} \text{ 上},\tag{17}$$

$$\frac{\partial}{\partial \boldsymbol{n}}\widetilde{H}^{[p]}+(2p-1)\kappa\widetilde{H}^{[p]}=(2p-1)H^{2p-2}F,\qquad \text{在 } \partial D\times\{x_3>0\}\times\{t>0\} \text{ 上},\tag{18}$$

$$\widetilde{H}^{[p]}(x_1,x_2,x_3,0)=0,\qquad \text{在 } R\times\{t=0\} \text{ 上}\tag{19}$$

的解，其中 $p\geqslant 1$ 是一个参数. 再令 $\sigma^{[p]}(\boldsymbol{x},t)=H^{2p-1}(x_1,x_2,t)\mathrm{e}^{-\gamma_1 x_3}$，$\gamma_1>0$. 显然，$\sigma^p$ 和 $\widetilde{H}^{[p]}$ 在 D 上有相同的边界条件. 所以由恒等式 $\int_R(\sigma^{[p]}-\widetilde{H}^{[p]})\Delta\widetilde{H}^{[p]}\mathrm{d}\boldsymbol{x}=0$，可得

$$\begin{aligned}\int_R\widetilde{H}^{[p]}_{,i}\widetilde{H}^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+(2p-1)\kappa\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]})^2\mathrm{d}l\mathrm{d}\xi&=\int_R\widetilde{H}^{[p]}_{,i}\sigma^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+(2p-1)\int_0^\infty\int_{\partial D_\xi}\widetilde{H}^{[p]}H^{2p-2}F\mathrm{d}l\mathrm{d}\xi+\\&\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}\sigma^{[p]}\widetilde{H}^{[p]}\mathrm{d}l\mathrm{d}\xi-(2p-1)\int_0^\infty\int_{\partial D_\xi}\sigma^{[p]}H^{2p-2}F\mathrm{d}l\mathrm{d}\xi\leqslant\\&\frac{1}{2}\int_R\widetilde{H}^{[p]}_{,i}\widetilde{H}^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+\frac{1}{2}\int_R\sigma^{[p]}_{,i}\sigma^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+\frac{1}{2}\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]})^2\mathrm{d}l\mathrm{d}\xi+\\&\frac{1}{2\kappa}(2p-1)\int_0^\infty\int_{\partial D_\xi}H^{4p-2}F^2\mathrm{d}l\mathrm{d}\xi+\frac{1}{2}\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\sigma^{[p]})^2\mathrm{d}l\mathrm{d}\xi,\end{aligned}$$

于是

$$\int_R\widetilde{H}^{[p]}_{,i}\widetilde{H}^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]})^2\mathrm{d}l\mathrm{d}\xi\leqslant k_3^{-1}\max\left\{1,\frac{1}{(2p-1)\kappa}\right\}^{-1}m_1(p,t),\tag{20}$$

其中

$$\begin{aligned}k_3^{-1}\max\left\{1,\frac{1}{(2p-1)\kappa}\right\}^{-1}m_1(p,t)=&\int_R\sigma^{[p]}_{,i}\sigma^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+\frac{1}{\kappa}(2p-1)\int_0^\infty\int_{\partial D_\xi}H^{4p-2}F^2\mathrm{d}l\mathrm{d}\xi+\\&\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\sigma^{[p]})^2\mathrm{d}l\mathrm{d}\xi.\end{aligned}\tag{21}$$

同理，由 $\int_R(\sigma^{[p]}_{,t}-\widetilde{H}^{[p]}_{,t})\Delta\widetilde{H}^{[p]}_{,t}\mathrm{d}\boldsymbol{x}=0$，可得

$$\int_R\widetilde{H}^{[p]}_{,tt}\widetilde{H}^{[p]}_{,tt}\mathrm{d}\boldsymbol{x}+\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]}_{,t})^2\mathrm{d}l\mathrm{d}\xi\leqslant k_3^{-1}\max\left\{1,\frac{1}{(2p-1)\kappa}\right\}^{-1}m_2(p,t),\tag{22}$$

其中

$$\begin{aligned}k_3^{-1}\max\left\{1,\frac{1}{(2p-1)\kappa}\right\}^{-1}m_2(p,t)=&\int_R\sigma^{[p]}_{,tt}\sigma^{[p]}_{,tt}\mathrm{d}\boldsymbol{x}+\frac{1}{\kappa}(2p-1)\int_0^\infty\int_{\partial D_\xi}H^{4p-2}F^2_{,t}\mathrm{d}l\mathrm{d}\xi+\\&\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\sigma^{[p]}_{,t})^2\mathrm{d}l\mathrm{d}\xi.\end{aligned}\tag{23}$$

利用引理 3 及式(20)，(22)，有

$$\begin{aligned}\int_R(\widetilde{H}^{[p]})^2\mathrm{d}\boldsymbol{x}\leqslant&\ k_3\max\left\{1,\frac{1}{(2p-1)\kappa}\right\}\times\\&\left(\int_R\widetilde{H}^{[p]}_{,i}\widetilde{H}^{[p]}_{,i}\mathrm{d}\boldsymbol{x}+\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]})^2\mathrm{d}l\mathrm{d}\xi\right)\leqslant m_1(p,t),\end{aligned}\tag{24}$$

$$\begin{aligned}\int_R(\widetilde{H}^{[p]}_{,t})^2\mathrm{d}\boldsymbol{x}\leqslant&\ k_3\max\left\{\frac{1}{(2p-1)\kappa},1\right\}\times\\&\left(\int_R\widetilde{H}^{[p]}_{,tt}\widetilde{H}^{[p]}_{,tt}\mathrm{d}\boldsymbol{x}+\kappa(2p-1)\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]}_{,t})^2\mathrm{d}l\mathrm{d}\xi\right)\leqslant m_2(p,t).\end{aligned}\tag{25}$$

利用式(24)，(25)，可得以下引理.

引理5 设 $\int_D f_3\,\mathrm{d}A=\int_D h_3\,\mathrm{d}A=0$, $\boldsymbol{f},\boldsymbol{h}\in L^2(R\times\{t>0\})$, $H\in L^\infty(R\times\{t>0\})$, 则

$$\frac{1}{2}\int_R T^2\,\mathrm{d}\boldsymbol{x}+\int_0^t\int_R T_{,i}T_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\leqslant n_1(t),$$

$$\int_R\int_0^t(b_1u_iu_i+b_2v_iv_i)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\leqslant n_2(t),$$

其中 $n_1(t)$和 $n_2(t)$是大于0且仅依赖于 t 的函数.

证明: 由

$$\int_0^t\int_R(T-\widetilde{H}^{[1]})\,(T_{,\eta}+c(u_i+v_i)T_{,i}-\Delta T)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta=0,$$

可得

$$\begin{aligned}&\frac{1}{2}\int_R T^2\,\mathrm{d}\boldsymbol{x}+\int_0^t\int_R T_{,i}T_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta=\\&\quad\int_R T\widetilde{H}^{[1]}\,\mathrm{d}\boldsymbol{x}-\int_0^t\int_R T\widetilde{H}^{[1]}_{,\eta}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_D f_3H^2\,\mathrm{d}A\mathrm{d}\eta+\\&\quad c\int_0^t\int_R(u_i+v_i)T_{,i}\widetilde{H}^{[1]}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_0^\infty\int_{\partial D_\xi}TF\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\\&\quad\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T\widetilde{H}^{[1]}\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta-\int_0^t\int_0^\infty\int_{\partial D_\xi}F\widetilde{H}^{[1]}\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta-\int_0^t\int_R T_{,i}\widetilde{H}^{[1]}_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\triangleq\sum_{i=1}^8\zeta_i.\end{aligned}\tag{26}$$

利用 Hölder 不等式、Young 不等式、引理3和式(24),(25)(取 $p=1$), 可得

$$\zeta_1\leqslant\left(\int_R T^2\,\mathrm{d}\boldsymbol{x}\int_R(\widetilde{H}^{[1]})^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\leqslant\frac{1}{4}\int_R T^2\,\mathrm{d}\boldsymbol{x}+\frac{1}{\lambda}m_1(1,t),\tag{27}$$

$$\begin{aligned}\zeta_2&\leqslant\left(\int_0^t\int_R T^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\int_0^t\int_R(\widetilde{H}^{[1]}_{,\eta})^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)^{1/2}\leqslant\\&\quad\sqrt{k_2}\max\left\{1,\frac{1}{\kappa}\right\}\left(\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\int_0^t\int_R T_{,\alpha}T_{,\alpha}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_R(\widetilde{H}^{[1]}_{,\eta})^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)^{1/2}\leqslant\\&\quad\frac{1}{8}\left(\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\int_0^t\int_R T_{,\alpha}T_{,\alpha}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)+2k_2\left[\max\left\{1,\frac{1}{\kappa}\right\}\right]^2\int_0^t m_2(1,\eta)\mathrm{d}\eta,\end{aligned}\tag{28}$$

$$\zeta_4\leqslant\frac{1}{8}\int_0^t\int_R T_{,i}T_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+2H_m^2\max\left\{\frac{1}{b_1},\frac{1}{b_2}\right\}\int_0^t\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta,\tag{29}$$

$$\zeta_5\leqslant\frac{1}{4}\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\frac{1}{\kappa}\int_0^t\int_0^\infty\int_{\partial D_\xi}F^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta,\tag{30}$$

$$\zeta_6\leqslant\frac{1}{4}\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[1]})^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta,\tag{31}$$

$$\zeta_8\leqslant\frac{1}{4}\int_0^t\int_R T_{,i}T_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_R\widetilde{H}^{[1]}_{,i}\widetilde{H}^{[1]}_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta,\tag{32}$$

其中 $H_m=\max\limits_{R\times\{t>0\}}|H|$. 把式(27)~(32)代入式(26), 并注意到 $\zeta_3,\zeta_7\leqslant0$ 再利用式(24),(25)(取 $p=1$), 可得

$$\frac{1}{2}\int_R T^2\,\mathrm{d}\boldsymbol{x}+\int_0^t\int_R T_{,i}T_{,i}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\leqslant m_3(t)+m_4\int_0^t\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta,\tag{33}$$

其中

$$m_3(t)=\max\left\{2,\frac{2}{\lambda}m_1(1,t)\right\}+2k_3\left(\max\left\{1,\frac{1}{\kappa}\right\}\right)^2m_1(1,t)+\frac{2}{\kappa}\int_0^t\int_0^\infty\int_{\partial D_\xi}F^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+2\int_0^t m_2(1,\eta)\mathrm{d}\eta,$$

$$m_4=4H_m^2\max\left\{\frac{1}{b_1},\frac{1}{b_2}\right\}.$$

利用式(1),(2), 有

$$a\int_{R_z}(u_i-v_i)(u_i-v_i)\mathrm{d}\boldsymbol{x}+\int_{R_z}(b_1u_iu_i+b_2v_iv_i)\,\mathrm{d}\boldsymbol{x}=$$

$$\int_{D_z} pu_3 \mathrm{d}A + \int_{D_z} qu_3 \mathrm{d}A + \int_{R_z} g_i T u_i \mathrm{d}\boldsymbol{x} + \int_{R_z} g_i T v_i \mathrm{d}\boldsymbol{x}. \tag{34}$$

注意到

$$\int_{D_z} u_3 \mathrm{d}A = \int_D u_3 \mathrm{d}A + \int_0^z \int_{D_\xi} \frac{\partial u_3}{\partial x_3} \mathrm{d}A \mathrm{d}\xi = \int_D f_3 \mathrm{d}A - \int_0^z \int_{D_\xi} u_{\alpha,\alpha} \mathrm{d}A \mathrm{d}\xi = \int_D f_3 \mathrm{d}A$$

和 $\int_D f_3 \mathrm{d}A = 0$，则 $\int_{D_z} u_3 \mathrm{d}A = 0$. 由引理 1 知，存在 $\psi = (\psi_1, \psi_2)$，满足

$$\psi_{\alpha,\alpha} = u_3, \quad \text{在 } D \text{ 上}; \qquad \psi_\alpha = 0, \quad \text{在 } \partial D \text{ 上}.$$

利用引理 1 和引理 2，可得

$$\int_{D_z} \psi_\alpha \psi_\alpha \mathrm{d}A \leqslant \frac{1}{\lambda} \int_{D_z} \psi_{\alpha,\beta} \psi_{\alpha,\beta} \mathrm{d}A \leqslant \frac{k_1}{\lambda} \int_{D_z} (\psi_{\alpha,\alpha})^2 \mathrm{d}A = \frac{k_1}{\lambda} \int_{D_z} u_3^2 \mathrm{d}A,$$

从而有

$$\begin{aligned} \int_{D_z} pu_3 \mathrm{d}A &= \int_{D_z} p\psi_{\alpha,\alpha} \mathrm{d}A = -\int_{D_z} p_{,\alpha} \psi_\alpha \mathrm{d}A = \int_{D_z} [b_1 u_\alpha - g_\alpha T + a(u_\alpha - v_\alpha)] \psi_\alpha \mathrm{d}A \leqslant \\ & b_1 \left(\int_{D_z} \psi_\alpha \psi_\alpha \mathrm{d}A \int_{D_z} u_\alpha u_\alpha \mathrm{d}A \right)^{1/2} + \left(\int_{D_z} \psi_\alpha \psi_\alpha \mathrm{d}A \int_{D_z} T^2 \mathrm{d}A \right)^{1/2} + \\ & a \left[\int_{D_z} (u_\alpha - v_\alpha)(u_\alpha - v_\alpha) \mathrm{d}A \right]^{1/2} \left(\int_{D_z} \psi_\alpha \psi_\alpha \mathrm{d}A \right)^{1/2} \leqslant \\ & \frac{b_1 \sqrt{k_1}}{2\sqrt{\lambda}} \int_{D_z} u_i u_i \mathrm{d}A + \frac{\sqrt{k_1}}{\sqrt{b_1 \lambda}} \left(\int_{D_z} T^2 \mathrm{d}A \mathrm{d}\eta + b_1 \int_{D_z} u_3^2 \mathrm{d}A \right) + \\ & \frac{\sqrt{ak_1}}{\sqrt{b_1 \lambda}} \left[b_1 \int_{D_z} u_3^2 \mathrm{d}A + a \int_{D_z} (u_\alpha - v_\alpha)(u_\alpha - v_\alpha) \mathrm{d}A \right]. \end{aligned} \tag{35}$$

同理，有

$$\begin{aligned} \int_{D_z} qu_3 \mathrm{d}A \leqslant & \frac{b_2 \sqrt{k_1}}{2\sqrt{\lambda}} \int_{D_z} v_i v_i \mathrm{d}A + \frac{\sqrt{k_1}}{\sqrt{b_2 \lambda}} \left(b_2 \int_{D_z} v_3^2 \mathrm{d}A + \int_{D_z} T^2 \mathrm{d}A \right) + \\ & \frac{\sqrt{ak_1}}{\sqrt{b_2 \lambda}} \left[b_2 \int_{D_z} v_3^2 \mathrm{d}A + a \int_{D_z} (u_\alpha - v_\alpha)(u_\alpha - v_\alpha) \mathrm{d}A \right]. \end{aligned} \tag{36}$$

利用 Hölder 不等式和 Young 不等式，有

$$\int_{R_z} g_i T u_i \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{2} b_1 \int_{R_z} u_i u_i \mathrm{d}\boldsymbol{x} + \frac{1}{2b_1} \int_{R_z} T^2 \mathrm{d}\boldsymbol{x}, \tag{37}$$

$$\int_{R_z} g_i T v_i \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{2} b_2 \int_{R_z} v_i v_i \mathrm{d}\boldsymbol{x} + \frac{1}{2b_2} \int_{R_z} T^2 \mathrm{d}\boldsymbol{x}. \tag{38}$$

把式(35)～(38)代入式(34)，可得

$$\begin{aligned} & a \int_{R_z} (u_i - v_i)(u_i - v_i) \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{R_z} (b_1 u_i u_i + b_2 v_i v_i) \mathrm{d}\boldsymbol{x} \leqslant \\ & \left(\frac{\sqrt{k_1}}{2\sqrt{\lambda}} + \frac{\sqrt{ak_1}}{\sqrt{b_1 \lambda}} + \frac{\sqrt{k_1}}{\sqrt{b_1 \lambda}} \right) \cdot b_1 \int_{D_z} u_i u_i \mathrm{d}A + \left(\frac{\sqrt{k_1}}{2\sqrt{\lambda}} + \frac{\sqrt{ak_1}}{\sqrt{b_2 \lambda}} + \frac{\sqrt{k_1}}{\sqrt{b_2 \lambda}} \right) \cdot b_2 \int_{D_z} v_i v_i \mathrm{d}A + \\ & \left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}} \right) \frac{\sqrt{ak_1}}{\sqrt{\lambda}} \cdot a \int_{D_z} (u_\alpha - v_\alpha)(u_\alpha - v_\alpha) \mathrm{d}A + \\ & \left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}} \right) \frac{\sqrt{k_1}}{\sqrt{\lambda}} \int_{D_z} T^2 \mathrm{d}A + \left(\frac{1}{2b_1} + \frac{1}{2b_2} \right) \int_{R_z} T^2 \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{39}$$

在式(39)中取 $z = 0$ 并利用式(4)，有

$$a \int_R (u_i - v_i)(u_i - v_i) \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_R (b_1 u_i u_i + b_2 v_i v_i) \mathrm{d}\boldsymbol{x} \leqslant \left(\frac{1}{2b_1} + \frac{1}{2b_2} \right) \int_R T^2 \mathrm{d}\boldsymbol{x} + m_5(t), \tag{40}$$

其中

$$m_5(t)=\left(\frac{\sqrt{k_1}}{2\sqrt{\lambda}}+\frac{\sqrt{ak_1}}{\sqrt{b_1\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_1\lambda}}\right)\cdot b_1\int_D f_i f_i\mathrm{d}A+\left(\frac{\sqrt{k_1}}{2\sqrt{\lambda}}+\frac{\sqrt{ak_1}}{\sqrt{b_2\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_2\lambda}}\right)\cdot b_2\int_D h_i h_i\mathrm{d}A+$$
$$\left(\frac{1}{\sqrt{b_1}}+\frac{1}{\sqrt{b_2}}\right)\frac{\sqrt{ak_1}}{\sqrt{\lambda}}\cdot a\int_D(f_\alpha-h_\alpha)(f_\alpha-h_\alpha)\mathrm{d}A+\left(\frac{\sqrt{k_1}}{\sqrt{b_1\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_2\lambda}}\right)\int_D H^2\mathrm{d}A.$$

结合式(33),(40),可得

$$\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\leqslant m_6(t)+m_7\int_0^t\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta,\tag{41}$$

其中

$$m_6(t)=m_5(t)+m_3(t)\left(\frac{1}{b_1}+\frac{1}{b_2}\right),\qquad m_7=m_4\left(\frac{1}{2b_1}+\frac{1}{2b_2}\right).$$

对式(41)从0到t积分,可得

$$\int_0^t\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\leqslant\int_0^t\exp\{m_7(t-\eta)\}m_6(\eta)\mathrm{d}\eta.\tag{42}$$

把式(42)代入式(33),可得

$$\frac{1}{2}\int_R T^2\mathrm{d}\boldsymbol{x}+\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\leqslant$$
$$m_4\int_0^t\exp\{m_7(t-\eta)\}m_6(\eta)\mathrm{d}\eta+m_3(t).\tag{43}$$

取

$$n_1(t)=m_4\int_0^t\exp\{m_7(t-\eta)\}m_6(\eta)\mathrm{d}\eta+m_3(t),\qquad n_2(t)=\int_0^t\exp\{m_7(t-\eta)\}m_6(\eta)\mathrm{d}\eta,$$

可得结论.

引理6 设$H\in L^\infty(R\times\{t>0\})$,$F\in L^\infty([0,\infty)\times\partial D\times\{t>0\})$,$\boldsymbol{f},\boldsymbol{h}\in L^2(R\times\{t>0\})$,$\int_D f_3\mathrm{d}A=\int_D h_3\mathrm{d}A=0$,则

$$\sup_{[0,t]}\|T\|_\infty\leqslant\max\{\|H\|_{L^\infty(R\times\{t>0\})},\|H_t\|_{L^\infty(R\times\{t>0\})},\|F\|_{L^\infty([0,\infty)\times\partial D\times\{t>0\})}\}\triangleq T_m.$$

证明:在方程(3)两边同时乘以$T^{2p-1}-\widetilde{H}^p$,并在$R\times[0,t]$上积分,可得

$$\frac{1}{2p}\int_R T^{2p}\mathrm{d}\boldsymbol{x}+\frac{2p-1}{p^2}\int_0^t\int_R(T^p)_{,i}(T^p)_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^{2p}\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta=$$
$$\int_R T\widetilde{H}^{[p]}\mathrm{d}\boldsymbol{x}-\int_0^t\int_R T\widetilde{H}^{[p]}_{,\eta}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta-\frac{1}{2p}\int_0^t\int_D f_3H^{2p}\mathrm{d}A\mathrm{d}\eta+$$
$$c\int_0^t\int_R(u_i+v_i)T_{,i}\widetilde{H}^{[p]}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_0^\infty\int_{\partial D_\xi}T^{2p-1}F\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T\widetilde{H}^{[p]}\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta-$$
$$\int_0^t\int_0^\infty\int_{\partial D_\xi}F\widetilde{H}^{[p]}\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta-\int_0^t\int_R T_{,i}\widetilde{H}^{[p]}_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\triangleq\sum_{i=1}^8\tau_i.\tag{44}$$

利用Hölder不等式和Young不等式、式(24),(25)及引理5,可得

$$\tau_1\leqslant\left[\int_R T^2\mathrm{d}\boldsymbol{x}\int_R(\widetilde{H}^{[p]})^2\mathrm{d}\boldsymbol{x}\right]^{1/2}\leqslant\frac{1}{\sqrt{\lambda}}\sqrt{2n_1(t)m_1(p,t)},\tag{45}$$

$$\tau_2\leqslant\left[\int_0^t\int_R T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\int_0^t\int_R(\widetilde{H}^{[p]}_{,\eta})^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right]^{1/2}\leqslant\sqrt{2\int_0^t n_1(\eta)\mathrm{d}\eta\int_0^t m_2(p,\eta)\mathrm{d}\eta},\tag{46}$$

$$\tau_4\leqslant H_m^{2p-1}\max\left\{\frac{1}{\sqrt{b_1}},\frac{1}{\sqrt{b_2}}\right\}\left[\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\int_0^t\int_R(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right]^{1/2}\leqslant$$
$$H_m^{2p-1}\max\left\{\frac{1}{\sqrt{b_1}},\frac{1}{\sqrt{b_2}}\right\}\sqrt{2n_1(t)\int_0^t n_2(\eta)\mathrm{d}\eta},\tag{47}$$

$$\tau_5\leqslant\kappa\int_0^t\int_0^\infty\int_{\partial D_\xi}T^{2p}\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\frac{1}{2p}\left(\frac{2p-1}{2p\kappa}\right)^{2p-1}\int_0^t\int_0^\infty\int_{\partial D_\xi}F^{2p}\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta,\tag{48}$$

$$\tau_6 \leqslant \kappa\left[\int_0^t\int_0^\infty\int_{\partial D_\xi} T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\int_0^t\int_0^\infty\int_{\partial D_\xi}(\widetilde{H}^{[p]})^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\right]^{1/2} \leqslant \sqrt{\frac{1}{2p-1}n_1(t)\int_0^t m_1(p,\eta)\mathrm{d}\eta}, \tag{49}$$

$$\tau_8 \leqslant \left(\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\int_0^t\int_R \widetilde{H}_{,i}^{[p]}\widetilde{H}_{,i}^{[p]}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)^{1/2} \leqslant \sqrt{n_1(t)\int_0^t m_1(p,\eta)\mathrm{d}\eta}. \tag{50}$$

把不等式(45)～(50)代入式(44)并注意到 $\tau_3,\tau_7\leqslant 0$，可得

$$\begin{aligned}\left(\int_R T^{2p}\mathrm{d}\boldsymbol{x}\right)^{1/(2p)} \leqslant&\left[\left(\frac{2p-1}{2p\kappa}\right)^{2p-1}\int_0^t\int_0^\infty\int_{\partial D_\xi}F^{2p}\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta+\frac{2p}{\sqrt{\lambda}}\sqrt{2n_1(t)m_1(p,t)}+\right.\\&\frac{2p}{\lambda}\sqrt{n_1(t)\int_0^t m_2(p,\eta)\mathrm{d}\eta}+2pH_m^{2p-1}\max\left\{\frac{1}{\sqrt{b_1}},\frac{1}{\sqrt{b_2}}\right\}\sqrt{2n_1(t)\int_0^t n_2(\eta)\mathrm{d}\eta}+\\&\left.2p\sqrt{\frac{1}{2p-1}n_1(t)\int_0^t m_1(p,\eta)\mathrm{d}\eta}+2p\sqrt{n_1(t)\int_0^t m_1(p,\eta)\mathrm{d}\eta}\right]^{1/(2p)}.\end{aligned} \tag{51}$$

在式(51)中令 $p\to\infty$并注意到 $m_1(p,t)$和 $m_2(p,t)$的定义(见式(21),(23))，即可完成证明.

3　空间衰减结果

首先，定义一个能量函数

$$\begin{aligned}\Phi(z,t)=&\frac{1}{4}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+a\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(u_i-v_i)(u_i-v_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&\frac{1}{2}\mathrm{e}^{-\delta t}\int_{R_z}(\xi-z)T^2\mathrm{d}\boldsymbol{x}+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta,\end{aligned} \tag{52}$$

其中 $\xi>z$，$\delta>0$. 对式(52)求导，可得

$$\begin{aligned}-\frac{\partial}{\partial z}\Phi(z,t)=&\frac{1}{4}\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(b_1u_iu_i+b_2v_iv_i)\mathrm{d}A\mathrm{d}\eta+a\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(u_i-v_i)(u_i-v_i)\mathrm{d}A\mathrm{d}\eta+\\&\frac{1}{2}\mathrm{e}^{-\delta t}\int_{R_z}T^2\mathrm{d}\boldsymbol{x}+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}T^2\,\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{53}$$

在式(53)中取 $z=0$ 并利用引理 5 和式(4)，可得

$$\begin{aligned}-\frac{\partial}{\partial z}\Phi(0,t)\leqslant&\frac{1}{4}\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(b_1f_if_i+b_2h_ih_i)\mathrm{d}A\mathrm{d}\eta+\\&a\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(f_i-h_i)(f_i-h_i)\mathrm{d}A\mathrm{d}\eta+n_1(t)\triangleq n_3(t).\end{aligned} \tag{54}$$

注意到

$$\int_{D_z}T^2\mathrm{d}A=2\int_z^\infty\int_{D_\xi}TT_{,3}\mathrm{d}A\mathrm{d}\xi\leqslant\gamma\int_{R_z}T^2\mathrm{d}\boldsymbol{x}+\frac{1}{\gamma}\int_{R_z}T_{,3}^2\mathrm{d}\boldsymbol{x},$$

其中 $\gamma>0$. 由式(39)可得

$$\begin{aligned}&\frac{1}{2}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+a\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(u_i-v_i)(u_i-v_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\leqslant\\&d_1\left[\frac{1}{4}\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(b_1u_iu_i+b_2v_iv_i)\mathrm{d}A\mathrm{d}\eta+a\int_0^t\int_{D_z}\mathrm{e}^{-\delta\eta}(u_i-v_i)(u_i-v_i)\mathrm{d}A\mathrm{d}\eta\right]+\\&\frac{d_2}{\gamma}\left(\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T_{,i}T_{,i}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\frac{1}{2}\delta\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right),\end{aligned} \tag{55}$$

其中

$$d_1=2\max\left\{\frac{\sqrt{k_1}}{2\sqrt{\lambda}}+\frac{\sqrt{ak_1}}{\sqrt{b_1\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_2\lambda}},\frac{\sqrt{k_1}}{2\sqrt{\lambda}}+\frac{\sqrt{ak_1}}{\sqrt{b_2\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_2\lambda}},\frac{\sqrt{ak_1}}{\sqrt{b_1\lambda}}+\frac{\sqrt{ak_1}}{\sqrt{b_2\lambda}}\right\},$$

$$d_2=\frac{\sqrt{k_1}}{\sqrt{b_1\lambda}}+\frac{\sqrt{k_1}}{\sqrt{b_2\lambda}},\quad d_3=\frac{1}{2\mu}+\frac{1}{2b_2},\quad \gamma=\frac{-d_3+\sqrt{d_3^2+32d_2^2\delta}}{4d_2}.$$

利用方程(3),(5),可得

$$\frac{1}{2}\mathrm{e}^{-\delta t}\int_{R_z}(\xi-z)T^2\mathrm{d}\boldsymbol{x}+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta=$$

$$\frac{1}{2}c\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(u_3+v_3)T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}TT_{,3}\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)TF\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\triangleq$$

$$I_1(z,t)+I_2(z,t)+I_3(z,t). \tag{56}$$

利用 Hölder 不等式和 Young 不等式以及引理 6,可得

$$I_1(z,t)\leqslant\frac{1}{4}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(b_1u_3^2+b_2v_3^2)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\frac{1}{4}c^2T_m^2\left(\frac{1}{b_1}+\frac{1}{b_2}\right)\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta, \tag{57}$$

$$I_2(z,t)\leqslant\frac{1}{\sqrt{2\delta}}\left(\frac{1}{2}\delta\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T_{,3}^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right), \tag{58}$$

$$I_3(z,t)\leqslant\left[\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)^2F^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\right]^{1/2}. \tag{59}$$

把不等式(57)~(59)代入式(56),可得

$$\frac{1}{2}\mathrm{e}^{-\delta t}\int_{R_z}(\xi-z)T^2\mathrm{d}\boldsymbol{x}+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\leqslant$$

$$d_4\left(\frac{1}{2}\delta\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}T_{,3}^2\mathrm{d}\boldsymbol{x}\mathrm{d}\eta\right)+$$

$$\left(\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)^2F^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\right)^{1/2}+$$

$$\frac{1}{4}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(b_1u_3^2+b_2v_3^2)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta, \tag{60}$$

其中 $d_4=\frac{1}{2\delta}c^2T_m^2\left(\frac{1}{d_1}+\frac{1}{d_2}\right)+\frac{1}{\sqrt{2\delta}}$. 将式(55)与式(60)相加并结合式(53),可得

$$\Phi(z,t)\leqslant\left[\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)^2F^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\left(-\frac{\partial}{\partial z}\Phi(z,t)\right)\right]^{1/2}+b_5\left(-\frac{\partial}{\partial z}\Phi(z,t)\right), \tag{61}$$

其中 $d_5=\max\left\{d_4+\frac{d_2}{\gamma},d_4\right\}$.

其次,定义一个新的辅助函数

$$\Psi(z,t)=\int_z^\infty\Phi(\zeta,t)\mathrm{d}\zeta. \tag{62}$$

对式(61)从 z 到∞积分并代入式(62),可得

$$\Psi(z,t)\leqslant d_6\left(-\frac{\partial}{\partial z}\Psi(z,t)\right)^{1/2}+d_5\left(-\frac{\partial}{\partial z}\Psi(z,t)\right),$$

即

$$\Psi(z,t)\leqslant\left(-\frac{\partial}{\partial z}\Psi(z,t)\right)^{1/2}\left[d_5\left(-\frac{\partial}{\partial z}\Psi(z,t)\right)^{1/2}+d_6\right], \tag{63}$$

其中 $d_6=\left(\frac{1}{6\kappa}\int_0^t\int_0^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}\xi^3F^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\right)^{1/2}$. 由于$\frac{\partial^2}{\partial z^2}\Psi(z,t)=\Phi(z,t)\geqslant0$,因此利用式(54),(61)可得

$$-\frac{\partial}{\partial z}\Psi(z,t)\leqslant\Phi(0,t)\leqslant d_5n_3(t)+\left(n_3(t)\int_0^t\int_0^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}\xi^2F^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\right)^{1/2}\triangleq n_4(t), \tag{64}$$

$$\Psi(0,t)\leqslant\sqrt{n_4(t)}(d_5\sqrt{n_4(t)}+d_6)\triangleq n_5(t). \tag{65}$$

于是式(63)可写为

$$\Psi(z,t)\leqslant d_7\left(-\frac{\partial}{\partial z}\Psi(z,t)\right)^{1/2}, \tag{66}$$

其中 $d_7=d_5\sqrt{n_4(t)}+d_6$. 对式(66)从 0 到 z 积分并利用式(65),可得

$$\Psi(z,t)\leqslant\frac{1}{\frac{1}{d_7^2}z+\frac{1}{n_5(t)}}. \tag{67}$$

令 $M=\min\left\{\frac{1}{d_7^2},\frac{1}{n_5(t)}\right\}$，由式(67)可得

$$\Psi(z,t)\leqslant\frac{1}{M(z+1)}. \tag{68}$$

另一方面，对式(52)从 z 到∞积分，可得

$$\begin{aligned}\Psi(z,t)=&\frac{1}{4}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+a\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)(u_i-v_i)(u_i-v_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&\mathrm{e}^{-\delta t}\int_{R_z}(\xi-z)^2T^2\mathrm{d}\boldsymbol{x}+2\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)^2\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&2\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)^2\ T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{69}$$

结合式(68)，(69)可得以下定理.

定理 1　设 $H\in L^\infty(R\times\{t>0\})$，$\boldsymbol{f},F\in L^\infty(\partial D\times[0,\infty)\times\{t>0\})$，$\boldsymbol{h}\in L^2(R\times\{t>0\})$，$\int_D f_3\mathrm{d}A=\int_D h_3\mathrm{d}A=0$，则

$$\begin{aligned}&\frac{1}{4}\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)(b_1u_iu_i+b_2v_iv_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+a\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)(u_i-v_i)(u_i-v_i)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&\mathrm{e}^{-\delta t}\int_{R_z}(\xi-z)^2T^2\mathrm{d}\boldsymbol{x}+2\int_0^t\int_{R_z}\mathrm{e}^{-\delta\eta}(\xi-z)^2\left(\frac{1}{2}\delta T^2+T_{,i}T_{,i}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\eta+\\&2\kappa\int_0^t\int_z^\infty\int_{\partial D_\xi}\mathrm{e}^{-\delta\eta}(\xi-z)^2\ T^2\mathrm{d}l\mathrm{d}\xi\mathrm{d}\eta\leqslant\frac{1}{M(z+1)}.\end{aligned}$$

注 1　定理 1 表明，当 $z\to\infty$时方程(1)-(7)的解代数式衰减，其衰减速度至少和 z^{-1}一样快. 这种衰减性结果可视为工程力学领域中的 Saint-Venant 原理型研究结果. 该原理目前已被推广到流体力学领域.

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