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Research article

Attouch-Wets convergence for closed sets in ⋆-metric spaces via scalarization

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Abstract: We developed an Attouch-Wets convergence theory for nonempty closed subsets of ⋆-metric spaces, associated with an admissible scalarization and a bounded testing family. Since the nonlinear triangle structure of a ⋆-metric space does not permit a direct transfer of the classical bounded-Hausdorff framework, we first introduced admissible scalarizations that convert bounded-region distance-profile comparisons into a workable additive setting. On this basis, we defined profile and truncated excess functionals associated with a scalarization, established equivalent formulations of the resulting convergence, and showed that it is pseudometrizable whenever the testing family has a countable cofinal subfamily. We then compared this convergence with the corresponding bornological convergences and with Wijsman convergence, and showed that the latter is strictly weaker in general, while equivalence holds under properness of the linearized metric. We also proved restriction and product results, together with relative compactness and sequential completeness theorems for the induced hyperspace structure. The examples showed that the theory is nontrivial in genuinely nonlinear ⋆-metric settings and that the main assumptions used in the comparison and compactness results are essential.

Keywords: ⋆-metric space; Attouch-Wets convergence; bornological convergence; hyperspace convergence; scalarization; Wijsman convergence
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1. Introduction

The asymptotic behavior of closed sets has long been studied through several mutually related but conceptually distinct hyperspace convergences. One of the most classical approaches is Wijsman

convergence, where a sequence of closed sets is analyzed through the pointwise convergence of the associated distance functions $x \longmapsto d(x, A_n)$. This profile-based viewpoint goes back to the pioneering work of Wijsman [14, 15] and has remained central in hyperspace topology and distance-based set convergence. A broader topological framework for such questions may be found in Beer's monograph [3].

Alongside this pointwise theory, a stronger line of development emerged from variational analysis and optimization. In that setting, one is often interested not merely in pointwise stabilization of distance profiles, but in a bounded-uniform form of convergence that controls closed sets simultaneously on prescribed regions of the ambient space. This is precisely the framework behind Attouch-Wets convergence, also called bounded Hausdorff convergence, and the corresponding bornological formulations [1, 2, 4, 5, 8]. The origin of this circle of ideas is closely related to Mosco convergence and epigraphical methods in convex and variational analysis [9, 12, 13]. In this way, Attouch-Wets-type convergence occupies a natural intermediate position between purely pointwise hyperspace modes and global Hausdorff control.

A different but complementary direction concerns the ambient geometry itself. In recent years, $\star$-metric spaces have provided a flexible nonlinear extension of the ordinary metric framework. Introduced by Khatami and Mirzavaziri [7], this class of spaces replaces the additive triangle inequality by a binary operation on $[0, \infty)$, called a t-definer. The subsequent work of He, Xie, and Yan [6] showed that $\star$-metric spaces still admit a substantial topological and sequential theory: the induced topology is Hausdorff, first countable, and metrizable, while completeness and boundedness properties remain meaningful in this generalized setting. Thus, although the ambient distance geometry is no longer additive, the class has enough structure to support a substantial convergence theory. Recent contributions to the broader theory of generalized metric-type structures also include results on octonion-valued b-metric spaces and on generalized ς-neutrosophic fuzzy metric spaces; see [10, 11]. These recent developments further support the need for a corresponding theory of closed-set convergence in generalized nonlinear metric-type environments.

Once a point-to-set distance is available in a $\star$-metric space, it is natural to ask how moving closed sets should be studied. A Wijsman-type approach is immediate and gives a first profile-level description of asymptotic behavior. However, pointwise control of distance profiles does not by itself guarantee bounded-region uniformity, nor does it capture the bornological features that are essential in approximation theory, variational problems, or local recovery of limiting sets. This gap suggests that an Attouch-Wets-type hyperspace theory should be developed in a form adapted to the nonlinear geometry of $\star$-metric spaces.

The main difficulty is that the classical bounded-Hausdorff framework is inherently additive, whereas the triangle structure in a $\star$-metric space is governed by a nonlinear operation. For this reason, a direct transplantation of the usual metric theory is not available. The approach adopted here is to pass first through a suitable scalarization of the $\star$-distance. Once such a scalarization is fixed, distance profiles can be compared on bounded testing regions within an ordinary additive scale, and this makes it possible to define an Attouch-Wets-type convergence associated with the chosen scalarization. For brevity, this will be called $(\phi, \mathcal{B})$-Attouch-Wets convergence. In this way, the present work may be viewed as a nonlinear extension of the classical Attouch-Wets framework carried out via scalarization.

The first objective of the paper is to identify a workable class of scalarizations and to show that they induce a coherent bounded-region hyperspace geometry. To this end, profile functionals and

truncated excess functionals associated with an admissible scalarization are introduced and their basic properties are established. This leads to the definition of $(\phi, \mathcal{B})$-Attouch-Wets convergence. Equivalent formulations are then derived, uniqueness of limits is proved, and pseudometrizability is obtained whenever the testing family has a countable cofinal subfamily.

A second objective is to place the new convergence within the surrounding landscape of set convergence. The comparison with the upper and lower bornological convergences induced by the same testing family shows that $(\phi, \mathcal{B})$-Attouch-Wets convergence is strictly stronger in general. Its relation to Wijsman convergence is also examined. While the new convergence always implies pointwise profile convergence, the converse fails in general. Under additional assumptions, however, especially when the scalarization is linearizing and the associated linearized metric is proper, the pointwise and bounded-uniform formulations coincide. This recovers, in the present setting, the familiar contrast between Wijsman and Attouch-Wets convergence from the classical metric theory.

A third objective is structural. Beyond the definition itself, a hyperspace convergence becomes useful only when it behaves well under the standard operations of set theory and geometry. For this reason, permanence under restriction to closed subspaces, preservation under finite products, and completeness-type consequences for the associated hyperspace structure are investigated. Under properness of the linearized metric, a relative compactness criterion and a sequential completeness theorem are also obtained. These results show that the present theory is not merely definable in $\star$-metric spaces, but also stable enough to support a genuine hyperspace theory.

The main difficulties of the paper are not merely formal. In a $\star$-metric space, the triangle inequality is governed by a nonlinear t-definer, so the classical bounded-Hausdorff arguments cannot be transferred verbatim. In particular, there is no a priori additive distance scale on which bounded-region profile comparisons can be carried out directly, and the passage from excess-type quantities to profile control requires additional care even on fixed testing regions. These difficulties are overcome by introducing admissible scalarizations, profile and truncated excess functionals adapted to bounded testing families, a cofinal reduction principle, and a global pseudometric under a countable cofinality assumption. This yields a theory that recovers the classical Attouch-Wets picture in the linearizing case, but also produces new bounded-region convergence phenomena in genuinely nonlinear $\star$-metric settings.

Although the paper is theoretical, the resulting convergence framework is relevant to settings in which feasible or admissible state sets evolve in ambient geometries that are not naturally additive. This includes, for example, moving constraint families in variational and optimization models, robustness questions for state sets under generalized dissimilarity measures, and uncertainty-driven frameworks built on nonclassical metric-type structures. Recent work on octonion-valued b-metric spaces and on generalized ς-neutrosophic fuzzy metric spaces indicates that such generalized geometries continue to arise in applied analysis; see [10, 11]. From this point of view, the present theory provides a bounded-region distance-profile tool for studying the stability of moving closed feasible sets in situations where global Hausdorff control is too strong and pointwise Wijsman-type control is too weak.

The examples included in the later sections clarify the sharpness of the results. They show that the framework remains meaningful in genuinely non-metric $\star$-spaces, that one-sided bornological control does not suffice to recover bounded-profile convergence, and that Wijsman convergence need not upgrade to Attouch-Wets-type convergence in the absence of properness. In particular, the reverse implications discussed in the paper fail in general once the corresponding additional assumptions are removed. Thus the present theory is both flexible and strictly stronger than the weaker profile-based

modes.

The paper is organized as follows. In Section 2, the necessary background on t-definers, $\star$-metric spaces, point-to-set distances, bounded testing families, and admissible scalarizations is recalled. Section 3 introduces the profile and truncated excess functionals used in the definition of $(\phi, \mathcal{B})$-Attouch-Wets convergence. In Section 4, this convergence is defined and its equivalent characterizations, uniqueness, and Cauchy description are established. Section 5 compares it with bornological and Wijsman convergence. Section 6 is devoted to restriction, product, compactness, and completeness results. Finally, Section 7 presents examples and counterexamples.

2. Preliminaries

This section collects the notation and background material needed in the following sections. The basic notions concerning t-definers and $\star$-metric spaces are recalled first. Bounded testing families and the scalarization principle used to pass from the nonlinear $\star$-distance to a workable bounded-region hyperspace theory are then introduced.

Throughout the paper, X denotes a nonempty set and $CL(X)$ denotes the family of all nonempty closed subsets of X with respect to the topology induced by the relevant $\star$-metric.

2.1. t-Definers and ⋆-metric spaces

The basic algebraic notion underlying the generalized triangle inequality is the following.

Definition 2.1. (t-Definer) [7] A mapping $\star : [0,\infty) \times [0,\infty) \to [0,\infty)$ is called a *t-definer* if, for all $a, b, c \in [0,\infty)$, the following conditions hold:

(1) $a \star b = b \star a$;

(2) $a \star (b \star c) = (a \star b) \star c$;

(3) if $a \leq b$, then $a \star c \leq b \star c$;

(4) $a \star 0 = a$;

(5) $\star$ is continuous in each variable.

The two canonical examples are the additive operation $a \star b = a + b$ and the maximum operation $a \star b = \max\{a, b\}$. They recover, respectively, the classical metric and ultrametric settings.

Definition 2.2. ($\star$-Metric) [7] Let $\star$ be a t-definer on $[0,\infty)$. A mapping $d_\star : X \times X \to [0,\infty)$ is called a *$\star$-metric* on X if, for all $x, y, z \in X$,

(1) $d_\star(x, y) = 0$ if and only if $x = y$;

(2) $d_\star(x, y) = d_\star(y, x)$;

(3) $d_\star(x, z) \leq d_\star(x, y) \star d_\star(y, z)$.

In this case, the pair $(X, d_\star)$ is called a $\star$-metric space.

For $x \in X$ and $r > 0$, we write $B_\star(x, r) = \{y \in X : d_\star(x, y) < r\}$ for the open $\star$-ball centered at x of radius r. The topology generated by these balls will be denoted by $\tau_{d_\star}$.

The following standard facts will be used repeatedly.

Theorem 2.3. *[6, 7] Let $(X, d_\star)$ be a $\star$-metric space. Then*

(1) the family of all $\star$-balls forms a base for a topology $\tau_{d_\star}$ on X;

(2) $(X, \tau_{d_\star})$ is Hausdorff and first countable;

(3) the topology $\tau_{d_\star}$ is metrizable;

(4) a sequence (x_n) converges to x in $\tau_{d_\star}$ if and only if $d_\star(x_n, x) \to 0$.

The next elementary consequence of the continuity of $\star$ at $(0, 0)$ will be used without further mention.

Lemma 2.4. *[6] For every $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$u < \eta,\ v < \eta \quad \Longrightarrow \quad u \star v < \varepsilon.$$

The standard sequential notions are recalled next.

Definition 2.5. ($\star$-Convergence and the $\star$-Cauchy sequence) [6] Let $(X, d_\star)$ be a $\star$-metric space and let (x_n) be a sequence in X.

(1) The sequence (x_n) is said to $\star$-*converge* to $x \in X$, written $x_n \xrightarrow{d_\star} x$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\star(x_n, x) < \varepsilon$ $(n \geq N)$.

(2) The sequence (x_n) is said to be a $\star$-*Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\star(x_n, x_m) < \varepsilon$ $(n, m \geq N)$.

The space $(X, d_\star)$ is called *complete* if every $\star$-Cauchy sequence in X is $\star$-convergent.

Example 2.6. On $[0, \infty)$, define $a \star b = (\sqrt{a} + \sqrt{b})^2$ and $d_\star(a, b) = (\sqrt{a} - \sqrt{b})^2$. Then $\star$ is a t-definer and $d_\star$ is a $\star$-metric. Moreover, $d_\star$ is not an ordinary metric, since the usual additive triangle inequality fails in general. This example will serve as a basic nonlinear model throughout the paper.

2.2. Point-to-set distance and bounded testing families

Let $(X, d_\star)$ be a fixed $\star$-metric space. For $x \in X$ and $A \subseteq X$, we define $d_\star(x, A) = \inf_{a \in A} d_\star(x, a)$, with the convention that A is always nonempty in the sequel.

This point-to-set distance is the natural analogue of the classical distance functional used throughout hyperspace theory. In particular, the profile approach underlying Wijsman, Attouch-Wets, and bornological convergence is based on the systematic use of distance, excess, and gap functionals; see [3, 8].

The usual closure characterization remains valid.

Lemma 2.7. *Let $A \subseteq X$ and let $x \in X$. Then $x \in \overline{A}$ if and only if $d_\star(x, A) = 0$. In particular, if $A \in CL(X)$, then $x \in A$ if and only if $d_\star(x, A) = 0$.*

Proof. If $x \in \overline{A}$, then for every $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $d_\star(x, a_\varepsilon) < \varepsilon$, hence $d_\star(x, A) = 0$.

Conversely, assume that $d_\star(x, A) = 0$. Then for every $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $d_\star(x, a_\varepsilon) < \varepsilon$, which means that every $\star$-ball centered at x meets A. Thus $x \in \overline{A}$. □

The families on which the local-uniform control will be tested are specified next.

Definition 2.8. A family $\mathcal{B}$ of nonempty subsets of X is called a *bounded testing family* if

(1) every $B \in \mathcal{B}$ is bounded in $(X, d_\star)$;

(2) whenever $\varnothing \neq E \subseteq B$ and $B \in \mathcal{B}$, one has $E \in \mathcal{B}$;

(3) whenever $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $B_1 \cup B_2 \subseteq B_3$;

(4) $\bigcup_{B\in\mathcal{B}} B = X$.

Thus a bounded testing family is simply a bornology consisting of bounded sets in the sense of hyperspace theory and bornological convergence. Typical examples are the families of finite subsets, bounded subsets, and compact subsets, whenever these form a cover of the ambient space; see, for example, [3, 8].

2.3. Admissible scalarizations

To build an Attouch-Wets-type theory in the nonlinear $\star$-metric setting, a way is needed to compare $\star$-distances through an ordinary additive scale. This is achieved by applying a scalar transform to the values of the distance functions.

Definition 2.9. Let $(X, d_\star)$ be a $\star$-metric space. A function $\phi : [0, \infty) \to [0, \infty)$ is called an admissible scalarization for $d_\star$ if the following conditions hold:

(1) ϕ is continuous and increasing;

(2) $\phi(0) = 0$, and $\phi(t) = 0$ if and only if $t = 0$;

(3) there exists a constant $C_\phi \geq 1$ such that $\phi(a \star b) \leq C_\phi\big(\phi(a) + \phi(b)\big)$ for all $a, b \geq 0$.

The third condition is the key compatibility requirement. It allows the nonlinear $\star$-triangle inequality to be converted into an additive estimate after applying ϕ.

Example 2.10. (1) If $a \star b = a + b$, then the identity map $\phi(t) = t$ is an admissible scalarization, with $C_\phi = 1$.

(2) If $a \star b = \max\{a, b\}$, then $\phi(t) = t$ is again admissible, since $\max\{a, b\} \leq a + b$.

(3) In the nonlinear model from Example 2.6, the function $\phi(t) = \sqrt{t}$ is admissible, and in fact satisfies $\phi(a \star b) = \phi(a) + \phi(b)$.

For later use, we record the basic Lipschitz-type consequence of admissibility.

Lemma 2.11. *Let ϕ be an admissible scalarization with constant C_ϕ. Then for all $x, y \in X$ and all $A \in CL(X)$,*

$$\phi\big(d_\star(x, A)\big) \leq C_\phi\Big(\phi\big(d_\star(x, y)\big) + \phi\big(d_\star(y, A)\big)\Big).$$

Consequently,

$$\phi\big(d_\star(x, A)\big) - C_\phi\phi\big(d_\star(y, A)\big) \leq C_\phi\phi\big(d_\star(x, y)\big).$$

Proof. Fix $a \in A$. By the $\star$-triangle inequality,

$$d_\star(x,a) \le d_\star(x,y) \star d_\star(y,a).$$

Applying ϕ and using Definition 2.9, we obtain

$$\phi\big(d_\star(x,a)\big) \le C_\phi\Big(\phi\big(d_\star(x,y)\big) + \phi\big(d_\star(y,a)\big)\Big).$$

Taking the infimum over $a \in A$ yields the first inequality. The second is immediate. □

The previous lemma shows that, after scalarization, distance profiles admit controlled local behavior on bounded regions. This will be the starting point for the hyperspace functionals introduced in the next section.

3. Scalarized truncated hyperspace functionals

This section introduces the bounded-region functionals used in the definition of $(\phi,\mathcal{B})$-Attouch-Wets convergence. Once an admissible scalarization is fixed, the point-to-set distance functions can be compared uniformly on bounded testing regions. This yields profile functionals and excess functionals adapted to the present setting.

Throughout this section, $(X,d_\star)$ is a fixed $\star$-metric space, ϕ is an admissible scalarization in the sense of Definition 2.9, and $\mathcal{B}$ is a bounded testing family on X.

The following lemma records that scalarized distance profiles are bounded on bounded testing sets.

Lemma 3.1. *Let $B \in \mathcal{B}$ and let $A \in CL(X)$. Then $\sup_{x\in B} \phi\big(d_\star(x,A)\big) < \infty$.*

Proof. Since B is bounded, there exist $x_B \in X$ and $M_B > 0$ such that $d_\star(x_B,x) \le M_B$ $(x \in B)$. Fix $a_0 \in A$. For each $x \in B$, the $\star$-triangle inequality gives

$$d_\star(x,a_0) \le d_\star(x,x_B) \star d_\star(x_B,a_0) \le M_B \star d_\star(x_B,a_0).$$

Since $d_\star(x,A) \le d_\star(x,a_0)$ and ϕ is increasing, we obtain

$$\phi\big(d_\star(x,A)\big) \le \phi\big(M_B \star d_\star(x_B,a_0)\big) \le C_\phi\Big(\phi(M_B) + \phi\big(d_\star(x_B,a_0)\big)\Big),$$

where C_ϕ is the constant from Definition 2.9. The right-hand side is independent of x, and the conclusion follows. □

3.1. Profile-based functionals

The profile functionals are introduced first by comparing the scalarized distance functions on a fixed bounded testing set.

Definition 3.2. Let $B \in \mathcal{B}$ and let $A, C \in CL(X)$. Define the *one-sided profile functional*

$$p^+_{\phi,B}(A,C) = \sup_{x\in B}\Big(\phi\big(d_\star(x,A)\big) - \phi\big(d_\star(x,C)\big)\Big)_+,$$

where $t_+ := \max\{t,0\}$ $(t \in \mathbb{R})$. We also define the *symmetric profile functional*

$$p_{\phi,B}(A,C) = \sup_{x\in B}\big|\phi\big(d_\star(x,A)\big) - \phi\big(d_\star(x,C)\big)\big|.$$

By Lemma 3.1, both functionals are finite for all $A, C \in CL(X)$ and all $B \in \mathcal{B}$.

The next proposition collects their elementary properties.

Proposition 3.3. *Let $B \in \mathcal{B}$ and let $A, C, D \in CL(X)$. Then:*

(1) $p_{\phi,B}(A, C) = \max\{p^+_{\phi,B}(A, C),\ p^+_{\phi,B}(C, A)\}$;

(2) $p_{\phi,B}(A, C) = p_{\phi,B}(C, A)$, $p_{\phi,B}(A, A) = 0$;

(3) $p_{\phi,B}(A, D) \le p_{\phi,B}(A, C) + p_{\phi,B}(C, D)$;

(4) if $B_1, B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then

$$p^+_{\phi,B_1}(A, C) \le p^+_{\phi,B_2}(A, C), \qquad p_{\phi,B_1}(A, C) \le p_{\phi,B_2}(A, C).$$

In particular, for each fixed $B \in \mathcal{B}$, the map $(A, C) \longmapsto p_{\phi,B}(A, C)$ is a pseudometric on $CL(X)$.

Proof. For each $x \in B$, we have

$$|u - v| = \max\{(u - v)_+, (v - u)_+\} \qquad (u, v \in \mathbb{R}),$$

and taking $u = \phi\big(d_\star(x, A)\big)$, $v = \phi\big(d_\star(x, C)\big)$, we obtain (1).

Part (2) is immediate from the definition. For (3), for every $x \in B$,

$$\big|\phi\big(d_\star(x, A)\big) - \phi\big(d_\star(x, D)\big)\big| \le \big|\phi\big(d_\star(x, A)\big) - \phi\big(d_\star(x, C)\big)\big| + \big|\phi\big(d_\star(x, C)\big) - \phi\big(d_\star(x, D)\big)\big|.$$

Taking the supremum over $x \in B$ yields the desired inequality. Part (4) follows directly from the monotonicity of the supremum with respect to the testing set. □

Thus the family $\{p_{\phi,B} : B \in \mathcal{B}\}$ already encodes a bounded-region hyperspace geometry. The associated convergence will be introduced in the next section.

3.2. Truncated excess and truncated Hausdorff functionals

The next definition introduces the one-sided quantities that measure how far one closed set is from another on a bounded testing region. These functionals are modeled on the classical excess and distance-profile quantities that play a central role in Wijsman, Attouch-Wets, and bornological hyperspace theories; see [3, 4, 8]. In the present scalarized $\star$-metric setting, they provide the natural one-sided building blocks for the bounded-region hyperspace convergence developed below.

Definition 3.4. Let $B \in \mathcal{B}$ and let $A, C \in CL(X)$. The *truncated excess* of A over C on B is defined by

$$\mathrm{ex}_{\phi,B}(A, C) = \sup_{a \in A \cap B} \phi\big(d_\star(a, C)\big),$$

with the convention that $\mathrm{ex}_{\phi,B}(A, C) = 0$ if $A \cap B = \varnothing$. The associated *truncated Hausdorff functional* is defined by

$$h_{\phi,B}(A, C) = \max\{\mathrm{ex}_{\phi,B}(A, C),\ \mathrm{ex}_{\phi,B}(C, A)\}.$$

These functionals are also finite on bounded testing regions.

Proposition 3.5. *Let $B \in \mathcal{B}$ and let $A, C \in CL(X)$. Then $\mathrm{ex}_{\phi,B}(A, C) < \infty$ and $h_{\phi,B}(A, C) < \infty$.*

Proof. If $A \cap B = \emptyset$, there is nothing to prove. Assume that $A \cap B \neq \emptyset$ and fix $c_0 \in C$. Since B is bounded, there exist $x_B \in X$ and $M_B > 0$ such that $d_\star(x_B, b) \leq M_B$ $(b \in B)$. For each $a \in A \cap B$, we have

$$d_\star(a, c_0) \leq d_\star(a, x_B) \star d_\star(x_B, c_0) \leq M_B \star d_\star(x_B, c_0).$$

As $d_\star(a, C) \leq d_\star(a, c_0)$ and ϕ is increasing,

$$\phi\big(d_\star(a, C)\big) \leq \phi\big(M_B \star d_\star(x_B, c_0)\big) \leq C_\phi\Big(\phi(M_B) + \phi\big(d_\star(x_B, c_0)\big)\Big),$$

which is independent of a. Taking the supremum over $a \in A \cap B$ yields $\mathrm{ex}_{\phi,B}(A, C) < \infty$. The finiteness of $h_{\phi,B}(A, C)$ follows immediately. □

The next proposition records the order-theoretic meaning of truncated excess.

Proposition 3.6. *Let $B \in \mathcal{B}$ and let $A, C \in CL(X)$. Then:*

(1) $\mathrm{ex}_{\phi,B}(A, C) = 0 \Longleftrightarrow A \cap B \subseteq C$;

(2) $h_{\phi,B}(A, C) = 0 \Longleftrightarrow A \cap B = C \cap B$.

Proof. If $\mathrm{ex}_{\phi,B}(A, C) = 0$, then for every $a \in A \cap B$, we have $\phi\big(d_\star(a, C)\big) = 0$. Since $\phi(t) = 0$ only for $t = 0$, it follows that $d_\star(a, C) = 0$, hence $a \in C$ by Lemma 2.7. Thus $A \cap B \subseteq C$.

Conversely, if $A \cap B \subseteq C$, then for every $a \in A \cap B$, $d_\star(a, C) = 0$, and therefore $\mathrm{ex}_{\phi,B}(A, C) = 0$. This proves (1).

For (2), if $h_{\phi,B}(A, C) = 0$, then both

$$\mathrm{ex}_{\phi,B}(A, C) = 0 \quad \text{and} \quad \mathrm{ex}_{\phi,B}(C, A) = 0,$$

so (1) yields $A \cap B \subseteq C$ and $C \cap B \subseteq A$, hence $A \cap B = C \cap B$. The converse is immediate. □

As in the profile case, the truncated excess functionals are monotone with respect to the testing region.

Proposition 3.7. *Let $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq B_2$, and let $A, C \in CL(X)$. Then*

$$\mathrm{ex}_{\phi,B_1}(A, C) \leq \mathrm{ex}_{\phi,B_2}(A, C), \qquad h_{\phi,B_1}(A, C) \leq h_{\phi,B_2}(A, C).$$

Proof. Since $A \cap B_1 \subseteq A \cap B_2$, the first inequality follows from the monotonicity of the supremum. The second is immediate from the definition of $h_{\phi,B}$. □

3.3. Comparison between profile and excess functionals

We next relate the profile functionals to the truncated excess quantities. The first comparison is immediate and already quite useful.

Proposition 3.8. *Let $B \in \mathcal{B}$ and let $A, C \in CL(X)$. Then*

$$\mathrm{ex}_{\phi,B}(A, C) \leq p_{\phi,B}(A, C), \qquad h_{\phi,B}(A, C) \leq p_{\phi,B}(A, C).$$

Proof. If $A \cap B = \varnothing$, then $\mathrm{ex}_{\phi,B}(A, C) = 0$ and the conclusion is trivial. Otherwise, let $a \in A \cap B$. Since $d_\star(a, A) = 0$, we have

$$\phi\big(d_\star(a, C)\big) = \big|\phi\big(d_\star(a, A)\big) - \phi\big(d_\star(a, C)\big)\big| \le p_{\phi,B}(A, C).$$

Taking the supremum over $a \in A \cap B$ gives $\mathrm{ex}_{\phi,B}(A, C) \le p_{\phi,B}(A, C)$. Exchanging the roles of A and C yields $\mathrm{ex}_{\phi,B}(C, A) \le p_{\phi,B}(A, C)$, and therefore

$$h_{\phi,B}(A, C) \le p_{\phi,B}(A, C).$$

□

Example 3.9. Let $X = \mathbb{R}$ be equipped with the ordinary metric, let $\star$ be the additive t-definer, let $\phi(t) = t$, and let $\mathcal{B}$ be the family of all bounded subsets of X. Take

$$B = \{0\}, \qquad A = \{2\}, \qquad C = \{3\}.$$

Then $A \cap B = C \cap B = \varnothing$, and therefore

$$\mathrm{ex}_{\phi,B}(A, C) = \mathrm{ex}_{\phi,B}(C, A) = h_{\phi,B}(A, C) = 0.$$

However,

$$p_{\phi,B}(A, C) = |d(0, A) - d(0, C)| = |2 - 3| = 1.$$

Thus there is no reverse estimate from the truncated Hausdorff functional to the profile functional on a fixed testing region.

Example 3.9 shows that truncated excess on a fixed testing region does not control the full profile functional. Accordingly, no reverse estimate of this type can hold in general on a fixed testing set. Any useful converse statement therefore requires additional structure, and in the present setting, this naturally leads to enlarged testing regions.

For later use, we introduce the standard notion of bounded enlargement.

Definition 3.10. Let $E \subseteq X$ be nonempty and let $r > 0$. The *open r-enlargement* of E is defined by

$$E^r_\star = \{x \in X : d_\star(x, E) < r\},$$

where $d_\star(x, E) = \inf_{y \in E} d_\star(x, y)$.

Remark 3.11. The functionals introduced above serve different but complementary purposes. The profile functional $p_{\phi,B}$ measures bounded-region uniform proximity of the scalarized distance profiles, whereas the truncated excess $\mathrm{ex}_{\phi,B}$ and the truncated Hausdorff functional $h_{\phi,B}$ encode one-sided and two-sided set-theoretic proximity on a bounded region. Example 3.9 shows that these two viewpoints are not equivalent on a fixed testing set. The passage to enlarged testing sets of the form $B^r_\star$ will therefore be essential in the equivalence results proved in the next section.

4. Attouch-Wets-type convergence and equivalent characterizations

This section introduces the central convergence notion of the paper. The asymptotic behavior of a sequence of closed sets is measured through the local-uniform convergence of their scalarized distance profiles on bounded testing regions. This is the natural analogue, in the present nonlinear setting, of the classical Attouch-Wets framework.

Throughout this section, $(X, d_\star)$ is a fixed $\star$-metric space, ϕ is an admissible scalarization, and $\mathcal{B}$ is a bounded testing family on X.

4.1. Definition of $(\phi, \mathcal{B})$-Attouch-Wets convergence

The basic definition is given first.

Definition 4.1. Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. The sequence (A_n) is said to *converge to A in the $(\phi, \mathcal{B})$-Attouch-Wets sense* if $p_{\phi,B}(A_n, A) \longrightarrow 0$ $(B \in \mathcal{B})$. In this case, we write $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.

By definition, for every bounded testing set $B \in \mathcal{B}$, the functions $x \longmapsto \phi\big(d_\star(x, A_n)\big)$ converge uniformly on B to $x \longmapsto \phi\big(d_\star(x, A)\big)$. Thus the convergence can be described either through the symmetric profile functional or through the simultaneous vanishing of the two one-sided profile functionals.

The one-sided profile functionals yield an immediate equivalent formulation.

Proposition 4.2. *Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. Then the following are equivalent:*

(1) $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$;

(2) for every $B \in \mathcal{B}$, $p^+_{\phi,B}(A_n, A) \longrightarrow 0$ and $p^+_{\phi,B}(A, A_n) \longrightarrow 0$;

(3) for every $B \in \mathcal{B}$, $\sup_{x\in B} \big|\phi\big(d_\star(x, A_n)\big) - \phi\big(d_\star(x, A)\big)\big| \longrightarrow 0$.

Proof. The equivalence of (1) and (3) is just Definition 4.1. The equivalence of (1) and (2) follows from Proposition 3.3(1), which gives

$$p_{\phi,B}(A_n, A) = \max\big\{p^+_{\phi,B}(A_n, A),\ p^+_{\phi,B}(A, A_n)\big\}$$

for every $B \in \mathcal{B}$ and every $n \in \mathbb{N}$. □

4.2. Cofinal subfamilies of the testing family

In practice, the testing family $\mathcal{B}$ may be replaced by a smaller subfamily that is still cofinal with respect to inclusion. The next proposition shows that the convergence is completely determined by such a family.

Definition 4.3. A subfamily $\mathcal{B}_0 \subseteq \mathcal{B}$ is called *cofinal in $\mathcal{B}$* if for every $B \in \mathcal{B}$, there exists $C \in \mathcal{B}_0$ such that $B \subseteq C$.

Proposition 4.4. *Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be cofinal in $\mathcal{B}$. For a sequence (A_n) in $CL(X)$ and a set $A \in CL(X)$, the following are equivalent:*

(1) $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$;

(2) for every $C \in \mathcal{B}_0$, $p_{\phi,C}(A_n, A) \longrightarrow 0$.

Proof. The implication (1)⇒(2) is immediate since $\mathcal{B}_0 \subseteq \mathcal{B}$.

Conversely, assume (2), and let $B \in \mathcal{B}$ be arbitrary. Choose $C \in \mathcal{B}_0$ such that $B \subseteq C$. By Proposition 3.3(4),

$$p_{\phi,B}(A_n, A) \leq p_{\phi,C}(A_n, A) \qquad (n \in \mathbb{N}).$$

Since the right-hand side tends to zero by assumption, the left-hand side also tends to zero. Hence $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$. □

This reduction is particularly useful when $\mathcal{B}$ has a countable cofinal subfamily. In that case, the convergence is determined by a single pseudometric.

4.3. A global pseudometric in the countable cofinal subfamily case

Assume now that $\mathcal{B}$ has a countable cofinal subfamily $\mathcal{B}_0 = \{B_m : m \in \mathbb{N}\}$. Without loss of generality, we may suppose that this subfamily is increasing, that is, $B_m \subseteq B_{m+1}$ $(m \in \mathbb{N})$, since each B_m may be replaced by a suitable cofinal enlargement of $\bigcup_{j=1}^{m} B_j$.

A global profile pseudometric is defined by summing the local pseudometrics p_{ϕ,B_m} with a standard bounded normalization.

Definition 4.5. Let $\{B_m : m \in \mathbb{N}\}$ be an increasing cofinal subfamily in $\mathcal{B}$. For $A, C \in CL(X)$, define

$$\delta_{\phi,\mathcal{B}}(A, C) = \sum_{m=1}^{\infty} 2^{-m} \frac{p_{\phi,B_m}(A, C)}{1 + p_{\phi,B_m}(A, C)}.$$

Since $0 \leq \frac{p_{\phi,B_m}(A,C)}{1+p_{\phi,B_m}(A,C)} \leq 1$, the series converges for all $A, C \in CL(X)$.

Proposition 4.6. *The map $\delta_{\phi,\mathcal{B}}$ is a pseudometric on $CL(X)$.*

Proof. Symmetry and the identity $\delta_{\phi,\mathcal{B}}(A, A) = 0$ follow at once from Proposition 3.3(2).

For the triangle inequality, let $A, C, D \in CL(X)$. By Proposition 3.3(3),

$$p_{\phi,B_m}(A, D) \leq p_{\phi,B_m}(A, C) + p_{\phi,B_m}(C, D) \qquad (m \in \mathbb{N}).$$

Now the function $\psi(t) = \frac{t}{1+t}$ $(t \geq 0)$ is increasing and subadditive, because

$$\psi(s + t) = \frac{s + t}{1 + s + t} \leq \frac{s}{1 + s} + \frac{t}{1 + t} = \psi(s) + \psi(t) \qquad (s, t \geq 0).$$

Hence

$$\frac{p_{\phi,B_m}(A, D)}{1 + p_{\phi,B_m}(A, D)} \leq \frac{p_{\phi,B_m}(A, C)}{1 + p_{\phi,B_m}(A, C)} + \frac{p_{\phi,B_m}(C, D)}{1 + p_{\phi,B_m}(C, D)}.$$

Multiplying by 2^{-m} and summing over m gives $\delta_{\phi,\mathcal{B}}(A, D) \leq \delta_{\phi,\mathcal{B}}(A, C) + \delta_{\phi,\mathcal{B}}(C, D)$. Thus $\delta_{\phi,\mathcal{B}}$ is a pseudometric. □

The following theorem gives the basic pseudometric characterization of the convergence in the countable cofinal subfamily case.

Theorem 4.7. *Assume that $\mathcal{B}$ has a countable cofinal subfamily $\{B_m : m \in \mathbb{N}\}$. Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. Then the following are equivalent:*

(1) $A_n \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} A$;

(2) $\delta_{\phi,\mathcal{B}}(A_n, A) \longrightarrow 0$;

(3) for every $m \in \mathbb{N}$, $p_{\phi,B_m}(A_n, A) \longrightarrow 0$.

Proof. The equivalence of (1) and (3) is exactly Proposition 4.4.

Assume (3). Then for each fixed m,

$$\frac{p_{\phi,B_m}(A_n, A)}{1 + p_{\phi,B_m}(A_n, A)} \longrightarrow 0.$$

Moreover, each term is bounded by 1. Therefore, by dominated convergence for the series, $\delta_{\phi,\mathcal{B}}(A_n, A) \longrightarrow 0$. Thus (3)⇒(2).

Now assume (2). Fix $m \in \mathbb{N}$. Since the mth summand in the definition of $\delta_{\phi,\mathcal{B}}(A_n, A)$ is nonnegative, we have

$$2^{-m} \frac{p_{\phi,B_m}(A_n, A)}{1 + p_{\phi,B_m}(A_n, A)} \leq \delta_{\phi,\mathcal{B}}(A_n, A).$$

Therefore $\frac{p_{\phi,B_m}(A_n,A)}{1+p_{\phi,B_m}(A_n,A)} \to 0$. Because

$$t_n \to 0 \Longleftrightarrow \frac{t_n}{1 + t_n} \to 0 \qquad (t_n \geq 0),$$

it follows that $p_{\phi,B_m}(A_n, A) \longrightarrow 0$. So (2)⇒(3), and the proof is complete. □

Thus, whenever the testing family has a countable cofinal subfamily, the $(\phi, \mathcal{B})$-Attouch-Wets convergence is pseudometrizable.

4.4. Uniqueness and Cauchy structure

The next result shows that the new convergence has unique limits.

Theorem 4.8. *Let (A_n) be a sequence in $CL(X)$. If $A_n \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} A$ and $A_n \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} C$, then $A = C$.*

Proof. Let $B \in \mathcal{B}$. By Proposition 3.3(3),

$$p_{\phi,B}(A, C) \leq p_{\phi,B}(A, A_n) + p_{\phi,B}(A_n, C) \ (n \in \mathbb{N}).$$

Since $p_{\phi,B}(A, A_n) = p_{\phi,B}(A_n, A) \to 0$ and $p_{\phi,B}(A_n, C) \to 0$, it follows that $p_{\phi,B}(A, C) = 0$ $(B \in \mathcal{B})$.

To prove $A \subseteq C$, let $x \in A$. Since $\bigcup \mathcal{B} = X$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x$. From $p_{\phi,B_x}(A, C) = 0$, one obtains

$$\phi\big(d_{\star}(x, A)\big) = \phi\big(d_{\star}(x, C)\big).$$

Because $x \in A$, one has $d_\star(x, A) = 0$, hence $\phi\big(d_\star(x, C)\big) = 0$. Since $\phi(t) = 0$ if and only if $t = 0$, it follows that $d_\star(x, C) = 0$, and therefore $x \in C$ by Lemma 2.7. Thus $A \subseteq C$.

The inclusion $C \subseteq A$ is obtained in the same way by interchanging the roles of A and C. Hence $A = C$. □

The associated Cauchy notion is introduced next.

Definition 4.9. A sequence (A_n) in $CL(X)$ is called $(\phi, \mathcal{B})$*-Attouch-Wets Cauchy* if for every $B \in \mathcal{B}$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $p_{\phi,B}(A_n, A_m) < \varepsilon$ $(n, m \geq N)$.

The one-sided and cofinal variants are again equivalent.

Proposition 4.10. *Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be cofinal. A sequence (A_n) in $CL(X)$ is $(\phi, \mathcal{B})$-Attouch-Wets Cauchy if and only if for every $C \in \mathcal{B}_0$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $p_{\phi,C}(A_n, A_m) < \varepsilon$ $(n, m \geq N)$.*

Proof. The proof is the same as that of Proposition 4.4, using Proposition 3.3(4). □

Whenever the testing family has a countable cofinal subfamily, the Cauchy property is again governed by the global pseudometric.

Theorem 4.11. *Assume that $\mathcal{B}$ has a countable cofinal subfamily $\{B_m : m \in \mathbb{N}\}$. For a sequence (A_n) in $CL(X)$, the following are equivalent:*

(1) (A_n) is $(\phi, \mathcal{B})$-Attouch-Wets Cauchy;

(2) for every $m \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $p_{\phi,B_m}(A_n, A_k) < \varepsilon$ $(n, k \geq N)$;

(3) (A_n) is Cauchy with respect to the pseudometric $\delta_{\phi,\mathcal{B}}$.

Proof. The equivalence of (1) and (2) follows from Proposition 4.10.

Assume (2). Fix $\varepsilon > 0$. Choose $M \in \mathbb{N}$ such that

$$\sum_{m>M} 2^{-m} < \frac{\varepsilon}{2}.$$

For each $1 \leq m \leq M$, condition (2) yields $N_m \in \mathbb{N}$ such that $p_{\phi,B_m}(A_n, A_k) < \eta$ $(n, k \geq N_m)$, where $\eta > 0$ will be chosen later. Let $N = \max\{N_1, \ldots, N_M\}$. Then for $n, k \geq N$,

$$\delta_{\phi,\mathcal{B}}(A_n, A_k) \leq \sum_{m=1}^{M} 2^{-m} \frac{\eta}{1+\eta} + \sum_{m>M} 2^{-m}.$$

Choosing η so that $\frac{\eta}{1+\eta} < \frac{\varepsilon}{2}$ gives $\delta_{\phi,\mathcal{B}}(A_n, A_k) < \varepsilon$. Hence (2)⇒(3).

Now assume (3). Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Since

$$2^{-m} \frac{p_{\phi,B_m}(A_n, A_k)}{1 + p_{\phi,B_m}(A_n, A_k)} \leq \delta_{\phi,\mathcal{B}}(A_n, A_k),$$

the Cauchy property in $\delta_{\phi,\mathcal{B}}$ implies

$$\frac{p_{\phi,B_m}(A_n, A_k)}{1 + p_{\phi,B_m}(A_n, A_k)} \longrightarrow 0 \qquad (n, k \to \infty).$$

Therefore $p_{\phi,B_m}(A_n, A_k) \to 0$ $(n, k \to \infty)$, which is (2). □

Thus the $(\phi, \mathcal{B})$-Attouch-Wets convergence and the associated Cauchy structure are described by the same pseudometric scheme as ordinary metric convergence whenever the testing family has a countable cofinal subfamily.

4.5. A first excess characterization

The profile formulation is the most natural starting point in the present framework. Still, it is useful to note that the new convergence automatically yields vanishing truncated excess on each testing region.

Proposition 4.12. *Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. If $A_n \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} A$, then for every $B \in \mathcal{B}$,*

$$\mathrm{ex}_{\phi,B}(A_n, A) \longrightarrow 0, \qquad \mathrm{ex}_{\phi,B}(A, A_n) \longrightarrow 0,$$

and consequently $h_{\phi,B}(A_n, A) \longrightarrow 0$.

Proof. This follows immediately from Proposition 3.8, since

$$\mathrm{ex}_{\phi,B}(A_n, A) \leq p_{\phi,B}(A_n, A), \qquad \mathrm{ex}_{\phi,B}(A, A_n) \leq p_{\phi,B}(A_n, A),$$

and $h_{\phi,B}(A_n, A) \leq p_{\phi,B}(A_n, A)$. □

The converse does not hold in general on a fixed testing set. Indeed, truncated excess only records the behavior of the two closed sets on their intersections with the testing region, whereas the profile functional compares the full scalarized distance functions on that region. If both sets stay outside a given bounded region, the truncated excess may vanish even though the distance profiles on that region are different. This observation is one of the basic reasons why the profile-based formulation is the correct primary notion for $(\phi, \mathcal{B})$-Attouch-Wets convergence.

Remark 4.13. The preceding proposition shows that Attouch-Wets-type convergence is stronger than the mere vanishing of truncated Hausdorff functionals on each fixed testing region. As shown already in Section 3, the reverse implication fails in general on a fixed testing set. In the classical theory, the passage from excess-based functionals to full bounded-profile control is closely tied to enlargements of the testing sets and to the structure of the underlying bornology. The corresponding comparison results in the present scalarized $\star$-metric setting will be taken up in the next section.

5. Comparison with bornological and Wijsman-type convergences

This section compares $(\phi, \mathcal{B})$-Attouch-Wets convergence with the natural bornological convergences induced by the testing family and with Wijsman convergence. The bornological modes are weaker in general, whereas Wijsman convergence becomes equivalent to it under additional compactness assumptions.

Throughout the section, $(X, d_\star)$ is a fixed $\star$-metric space, ϕ is an admissible scalarization, and $\mathcal{B}$ is a bounded testing family on X.

5.1. Upper and lower bornological convergences

The truncated excess functionals from Section 3 lead naturally to one-sided and two-sided bornological convergence modes.

Definition 5.1. Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$.

(1) The sequence (A_n) is said to *lower bornologically converge* to A with respect to $(\phi, \mathcal{B})$ if $\mathrm{ex}_{\phi,B}(A, A_n) \longrightarrow 0$ for every $B \in \mathcal{B}$. In this case, we write $A_n \xrightarrow{\mathcal{B}^-_\phi} A$.

(2) The sequence (A_n) is said to *upper bornologically converge* to A with respect to $(\phi, \mathcal{B})$ if $\mathrm{ex}_{\phi,B}(A_n, A) \longrightarrow 0$ for every $B \in \mathcal{B}$. In this case, we write $A_n \xrightarrow{\mathcal{B}^+_\phi} A$.

(3) The sequence (A_n) is said to *bornologically converge* to A with respect to $(\phi, \mathcal{B})$ if both lower and upper bornological convergence hold, or equivalently, if $h_{\phi,B}(A_n, A) \longrightarrow 0$ for every $B \in \mathcal{B}$. In this case, we write $A_n \xrightarrow{\mathcal{B}_\phi} A$.

This terminology agrees with the usual one-sided bornological viewpoint: the lower mode controls the approximation of A by A_n on each testing region, whereas the upper mode prevents the approximating sets from developing bounded pieces that remain separated from the limit set.

The first comparison is immediate but fundamental.

Theorem 5.2. *Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. If $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$, then $A_n \xrightarrow{\mathcal{B}^-_\phi} A$, $A_n \xrightarrow{\mathcal{B}^+_\phi} A$, and hence $A_n \xrightarrow{\mathcal{B}_\phi} A$.*

Proof. This is exactly Proposition 4.12. □

Thus $(\phi, \mathcal{B})$-Attouch-Wets convergence is always stronger than the corresponding bornological modes. No reverse implication of this type can be expected in general. In fact, even the one-sided bornological convergences do not imply $(\phi, \mathcal{B})$-Attouch-Wets convergence, as the next two examples show.

Example 5.3. Let

$$X = \mathbb{R}, \qquad a \star b = a + b, \qquad \phi(t) = t,$$

and let $\mathcal{B}$ be the family of all bounded subsets of $\mathbb{R}$. Define $A_n = \{n\}$ and $A = \{0\}$. Then

$$A_n \xrightarrow{\mathcal{B}^+_\phi} A,$$

but $A_n \not\xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.

Proof. Let $B \in \mathcal{B}$ be fixed. Since B is bounded, there exists $M > 0$ such that $B \subseteq [-M, M]$. For $n > M$, one has $A_n \cap B = \varnothing$, and therefore

$$\mathrm{ex}_{\phi,B}(A_n, A) = 0.$$

Hence $A_n \xrightarrow{\mathcal{B}^+_\phi} A$.

On the other hand, taking $B = \{0\}$ gives

$$p_{\phi,\{0\}}(A_n, A) = |d(0, \{n\}) - d(0, \{0\})| = n,$$

which does not tend to zero. Therefore

$$A_n \overset{AW^{\star}_{\phi,\mathcal{B}}}{\not\longrightarrow} A.$$

□

Example 5.4. Let

$$X = \mathbb{R}, \qquad a \star b = a + b, \qquad \phi(t) = t,$$

and let $\mathcal{B}$ be the family of all bounded subsets of $\mathbb{R}$. Define $A_n = [0, \infty)$ and $A = \{0\}$. Then

$$A_n \overset{\mathcal{B}^{-}_{\phi}}{\longrightarrow} A,$$

but $A_n \overset{AW^{\star}_{\phi,\mathcal{B}}}{\not\longrightarrow} A$.

Proof. For every bounded set $B \in \mathcal{B}$, one has $A \cap B \subseteq A_n$ $(n \in \mathbb{N})$, because $0 \in A_n$ for every n. Hence

$$\mathrm{ex}_{\phi,B}(A, A_n) = 0 \ (n \in \mathbb{N}),$$

so $A_n \overset{\mathcal{B}^{-}_{\phi}}{\longrightarrow} A$.

However, for $B = [0, 1]$,

$$p_{\phi,B}(A_n, A) = \sup_{x\in[0,1]} |d(x, [0, \infty)) - d(x, \{0\})|.$$

Since $d(x, [0, \infty)) = 0$ and $d(x, \{0\}) = x$ for $x \in [0, 1]$, it follows that

$$p_{\phi,B}(A_n, A) = \sup_{x\in[0,1]} x = 1.$$

Therefore $A_n \overset{AW^{\star}_{\phi,\mathcal{B}}}{\not\longrightarrow} A$. □

Remark 5.5. The preceding examples show that one-sided set-theoretic control does not recover the bounded-region profile convergence used in the definition of $(\phi, \mathcal{B})$-Attouch-Wets convergence. This is why the profile-based formulation, rather than the bornological one, must be treated as the primary notion in the present theory.

5.2. The linearizing case and recovery of the classical metric framework

The scalarized theory takes a simpler form when the scalarization converts the $\star$-triangle inequality into an ordinary additive one.

Definition 5.6. An admissible scalarization ϕ is called *linearizing* if it is strictly increasing and satisfies

$$\phi(a \star b) = \phi(a) + \phi(b) \qquad \text{for all } a, b \geq 0.$$

In that case, the scalarized $\star$-distance becomes an ordinary metric.

Proposition 5.7. *Assume that ϕ is linearizing, and define $\rho_\phi(x,y) := \phi\big(d_\star(x,y)\big)$ $(x,y \in X)$. Then ρ_ϕ is a metric on X. Moreover, for every $x \in X$ and every $A \in CL(X)$,*

$$\rho_\phi(x,A) = \phi\big(d_\star(x,A)\big),$$

where $\rho_\phi(x,A) := \inf_{a\in A} \rho_\phi(x,a)$.

Proof. Since ϕ is strictly increasing and $\phi(t) = 0$ if and only if $t = 0$,

$$\rho_\phi(x,y) = 0 \Longleftrightarrow d_\star(x,y) = 0 \Longleftrightarrow x = y.$$

Symmetry is immediate. For the triangle inequality,

$$\rho_\phi(x,z) = \phi\big(d_\star(x,z)\big) \le \phi\big(d_\star(x,y) \star d_\star(y,z)\big) = \phi\big(d_\star(x,y)\big) + \phi\big(d_\star(y,z)\big) = \rho_\phi(x,y) + \rho_\phi(y,z).$$

Hence ρ_ϕ is a metric.

Finally, since ϕ is continuous and increasing,

$$\rho_\phi(x,A) = \inf_{a\in A} \phi\big(d_\star(x,a)\big) = \phi\left(\inf_{a\in A} d_\star(x,a)\right) = \phi\big(d_\star(x,A)\big).$$

□

So, in the linearizing case, the profile functionals from Section 3 are simply the ordinary bounded-uniform profile functionals associated with the metric ρ_ϕ.

Theorem 5.8. *Assume that ϕ is linearizing. Then for all $B \in \mathcal{B}$ and all $A, C \in CL(X)$,*

$$p_{\phi,B}(A,C) = \sup_{x\in B}\big|\rho_\phi(x,A) - \rho_\phi(x,C)\big|, \qquad \mathrm{ex}_{\phi,B}(A,C) = \sup_{a\in A\cap B} \rho_\phi(a,C),$$

and

$$h_{\phi,B}(A,C) = \max\left\{\sup_{a\in A\cap B} \rho_\phi(a,C),\ \sup_{c\in C\cap B} \rho_\phi(c,A)\right\}.$$

Consequently, for a sequence (A_n) in $CL(X)$ and a set $A \in CL(X)$, $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$ if and only if the distance functions $x \longmapsto \rho_\phi(x,A_n)$ converge to $x \longmapsto \rho_\phi(x,A)$ uniformly on every $B \in \mathcal{B}$.

Proof. The identities follow directly from Proposition 5.7 and the definitions of the corresponding functionals. The final assertion is therefore an immediate reformulation of Definition 4.1 in terms of the metric ρ_ϕ. □

In particular, if $\mathcal{B}$ is the family of all ρ_ϕ-bounded subsets of X, $(\phi,\mathcal{B})$-Attouch-Wets convergence is precisely the usual bounded-set Attouch-Wets convergence on the metric space (X,ρ_ϕ); similarly,

$$\xrightarrow{\mathcal{B}^-_\phi}, \qquad \xrightarrow{\mathcal{B}^+_\phi}, \qquad \xrightarrow{\mathcal{B}_\phi}$$

recover the corresponding one-sided and two-sided bornological convergences in the metric setting [1, 4, 8].

5.3. Comparison with Wijsman-type convergence

The pointwise profile theory is considered next.

Definition 5.9. Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. The sequence (A_n) is said to *converge to A in the scalarized Wijsman sense* if

$$\phi\big(d_\star(x, A_n)\big) \longrightarrow \phi\big(d_\star(x, A)\big) \ (x \in X).$$

In this case, we write $A_n \xrightarrow{W^\star_\phi} A$.

If ϕ is strictly increasing, this is equivalent to the pointwise convergence of the original distance profiles $d_\star(x, A_n) \to d_\star(x, A)$ $(x \in X)$, so Definition 5.9 is then exactly the usual Wijsman-type convergence adapted to the $\star$-metric setting.

The relation with the Attouch-Wets-type convergence is immediate.

Theorem 5.10. *Let (A_n) be a sequence in $CL(X)$ and let $A \in CL(X)$. If $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$, then $A_n \xrightarrow{W^\star_\phi} A$.*

Proof. Fix $x \in X$. Since $\mathcal{B}$ is a bounded testing family and covers X, there exists $B_x \in \mathcal{B}$ with $x \in B_x$. Because $\mathcal{B}$ is hereditary, $\{x\} \in \mathcal{B}$. Hence

$$\big|\phi\big(d_\star(x, A_n)\big) - \phi\big(d_\star(x, A)\big)\big| = p_{\phi,\{x\}}(A_n, A) \le p_{\phi,B_x}(A_n, A) \longrightarrow 0.$$

Thus $\phi\big(d_\star(x, A_n)\big) \to \phi\big(d_\star(x, A)\big)$ for every $x \in X$. □

Thus $(\phi, \mathcal{B})$-Attouch-Wets convergence always implies Wijsman convergence.

The converse is false in general, even in the ordinary metric setting. The following example shows that pointwise profile convergence need not control the distance profiles uniformly on bounded regions.

Example 5.11. Let $X = \ell^2$ be the real Hilbert space of square-summable sequences, equipped with the usual metric, let $\star$ be the additive t-definer, let $\phi(t) = t$, and let $\mathcal{B}$ be the family of all bounded subsets of ℓ^2.

For each $n \in \mathbb{N}$, define

$$A_n := \{x = (x_k)_{k\ge1} \in \ell^2 : x_n = 0\},$$

and let $A := \ell^2$. Then each A_n is a closed linear subspace of codimension one, and for every $x = (x_k) \in \ell^2$, we have

$$d(x, A_n) = |x_n| \longrightarrow 0 = d(x, A),$$

because $x_n \to 0$. Hence $A_n \xrightarrow{W^\star_\phi} A$.

On the other hand, if

$$B := \{x \in \ell^2 : \|x\| \le 1\},$$

then

$$p_{\phi,B}(A_n, A) = \sup_{\|x\|\le1} |d(x, A_n) - d(x, A)| = \sup_{\|x\|\le1} d(x, A_n).$$

Since the nth unit vector e_n belongs to B and satisfies $d(e_n, A_n) = 1$, it follows that

$$p_{\phi,B}(A_n, A) \ge 1 \qquad (n \in \mathbb{N}).$$

Therefore $A_n \not\xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.

Thus Wijsman convergence is, in general, strictly weaker than $(\phi, \mathcal{B})$-Attouch-Wets convergence.

Example 5.11 shows that the reverse implication fails in general. The next theorem shows that properness of the linearized metric is the key additional assumption under which Wijsman convergence and $(\phi, \mathcal{B})$-Attouch-Wets convergence become equivalent.

Theorem 5.12. *Assume that ϕ is linearizing and let $\rho_\phi(x, y) = \phi\big(d_\star(x, y)\big)$. Suppose that (X, ρ_ϕ) is proper, and let $\mathcal{B}$ be the family of all ρ_ϕ-bounded subsets of X. If $A_n \xrightarrow{W^\star_\phi} A$, then $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.*

Proof. Fix $B \in \mathcal{B}$. Since B is ρ_ϕ-bounded and (X, ρ_ϕ) is proper, the closure $\overline{B}^{\rho_\phi}$ is compact in (X, ρ_ϕ).

For each $n \in \mathbb{N}$, define

$$f_n(x) := \rho_\phi(x, A_n), \qquad f(x) := \rho_\phi(x, A) \qquad (x \in X).$$

By Proposition 5.7,

$$f_n(x) = \phi\big(d_\star(x, A_n)\big), \qquad f(x) = \phi\big(d_\star(x, A)\big).$$

The assumption $A_n \xrightarrow{W^\star_\phi} A$ therefore says exactly that $f_n(x) \longrightarrow f(x)$ for every $x \in X$.

Moreover, each f_n is 1-Lipschitz on (X, ρ_ϕ), because for all $x, y \in X$,

$$|f_n(x) - f_n(y)| = |\rho_\phi(x, A_n) - \rho_\phi(y, A_n)| \le \rho_\phi(x, y),$$

and the same is true of f.

We claim that $f_n \to f$ uniformly on $\overline{B}^{\rho_\phi}$. Let $\varepsilon > 0$ be given. By compactness, there exist points $x_1, \dots, x_M \in \overline{B}^{\rho_\phi}$ such that

$$\overline{B}^{\rho_\phi} \subseteq \bigcup_{j=1}^{M} B_{\rho_\phi}\left(x_j, \frac{\varepsilon}{3}\right).$$

For each j, since $f_n(x_j) \to f(x_j)$, there exists N_j such that

$$|f_n(x_j) - f(x_j)| < \frac{\varepsilon}{3} \ (n \ge N_j).$$

Let $N = \max\{N_1, \dots, N_M\}$. If $n \ge N$ and $x \in \overline{B}^{\rho_\phi}$, choose j such that $\rho_\phi(x, x_j) < \frac{\varepsilon}{3}$. Then, using the Lipschitz property,

$$|f_n(x) - f(x)| \le |f_n(x) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence

$$\sup_{x \in B} |f_n(x) - f(x)| \le \sup_{x \in \overline{B}^{\rho_\phi}} |f_n(x) - f(x)| \longrightarrow 0.$$

By Theorem 5.8, $p_{\phi,B}(A_n, A) \to 0$. Since $B \in \mathcal{B}$ was arbitrary, we conclude that $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$. □

Remark 5.13. Example 5.11 also shows that the properness assumption in the previous equivalence theorem cannot be omitted in general.

Combining Theorems 5.10 and 5.12, we obtain the following.

Corollary 5.14. *Assume that ϕ is linearizing, that (X, ρ_ϕ) is proper, and that $\mathcal{B}$ is the family of all ρ_ϕ-bounded subsets of X. Then, for sequences of nonempty closed sets, $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$ if and only if $A_n \xrightarrow{W^\star_\phi} A$.*

Remark 5.15. The results of this section clarify the position of $(\phi, \mathcal{B})$-Attouch-Wets convergence among the surrounding hyperspace modes. It is always stronger than the upper and lower bornological convergences induced by the same testing family, and it is also stronger than Wijsman convergence in general. On the other hand, when a linearizing scalarization is available and the associated linearized metric is proper, the pointwise and bounded-region formulations become equivalent. This is consistent with the classical metric theory and shows that the present framework reproduces the expected Attouch-Wets behavior in the scalarized $\star$-metric setting.

6. Completeness, restriction, and products

This section addresses the structural side of the theory. The purpose is to show that the present framework is stable under the basic constructions expected in hyperspace analysis. Restriction to closed subspaces is studied first, followed by a permanence result for products. Finally, under a properness assumption on the linearized metric, a relative compactness criterion and a sequential completeness theorem are established for the associated hyperspace convergence.

6.1. Restriction to closed subspaces

The simplest permanence property is considered first.

Definition 6.1. Let $(X, d_\star)$ be a $\star$-metric space, let $Y \subseteq X$ be nonempty, and let $\mathcal{B}$ be a bounded testing family on X. We define the *restricted testing family* on Y by

$$\mathcal{B}|_Y = \{\, B \cap Y : \; B \in \mathcal{B},\; B \cap Y \neq \varnothing \,\}.$$

If Y is equipped with the restricted $\star$-metric $d^Y_\star := d_\star|_{Y\times Y}$, then $\mathcal{B}|_Y$ is clearly a bounded testing family on Y.

Theorem 6.2. *Let $(X, d_\star)$ be a $\star$-metric space, let $Y \subseteq X$ be closed, and let $\mathcal{B}$ be a bounded testing family on X. Suppose that $A_n, A \in CL(Y)$ $(n \in \mathbb{N})$. If $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$ in X, then $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}|_Y}} A$ in Y.*

Proof. Since Y is closed in X and each A_n and A is closed in Y, they are also closed in X. Let $C \in \mathcal{B}|_Y$. By definition, there exists $B \in \mathcal{B}$ such that $C = B \cap Y$. For every $y \in Y$, the point-to-set distances computed in Y and in X agree, because the underlying set over which the infimum is taken is the same:

$$d^Y_\star(y, A_n) = d_\star(y, A_n), \qquad d^Y_\star(y, A) = d_\star(y, A).$$

Therefore

$$p^Y_{\phi,C}(A_n, A) = \sup_{y\in C} \left|\phi\big(d^Y_\star(y, A_n)\big) - \phi\big(d^Y_\star(y, A)\big)\right| = \sup_{y\in C} \left|\phi\big(d_\star(y, A_n)\big) - \phi\big(d_\star(y, A)\big)\right|.$$

Since $C \subseteq B$, we obtain

$$p^{Y}_{\phi,C}(A_n, A) \le p^{X}_{\phi,B}(A_n, A).$$

The right-hand side tends to zero by the assumed ambient convergence. As $C \in \mathcal{B}|_Y$ was arbitrary, the conclusion follows. □

Thus $(\phi, \mathcal{B})$-Attouch-Wets convergence always passes to closed subspaces.

The same argument yields the corresponding Cauchy permanence.

Corollary 6.3. *Under the assumptions of Theorem* 6.2*, if* (A_n) *is* $(\phi, \mathcal{B})$*-Attouch-Wets Cauchy in* X*, then it is* $(\phi, \mathcal{B}|_Y)$*-Attouch-Wets Cauchy in* Y*.*

Proof. For each $C \in \mathcal{B}|_Y$, choose $B \in \mathcal{B}$ with $C = B \cap Y$. Then

$$p^{Y}_{\phi,C}(A_n, A_m) \le p^{X}_{\phi,B}(A_n, A_m) \ (n, m \in \mathbb{N}),$$

and the conclusion follows directly from Definition 4.9. □

6.2. Products

The corresponding permanence result for finite products is established next. Throughout this part, the scalarization is assumed to be linearizing.

Let $(X_1, d_{\star,1})$ and $(X_2, d_{\star,2})$ be $\star$-metric spaces for the same t-definer, and assume that ϕ is a linearizing scalarization for both spaces. Define

$$\rho_i(x_i, y_i) := \phi\big(d_{\star,i}(x_i, y_i)\big) \ (x_i, y_i \in X_i, \ i = 1, 2).$$

By Proposition 5.7, each ρ_i is an ordinary metric on X_i.

On the product $X_1 \times X_2$, consider the max metric

$$\rho_\times\big((x_1, x_2), (y_1, y_2)\big) := \max\{\rho_1(x_1, y_1), \ \rho_2(x_2, y_2)\}.$$

Let $\mathcal{B}_i$ be bounded testing families on X_i, consisting of ρ_i-bounded sets. We denote by

$$\mathcal{B}_1 \boxtimes \mathcal{B}_2 = \{ \, B_1 \times B_2 : \ B_i \in \mathcal{B}_i \, \}$$

the corresponding rectangular testing family on $X_1 \times X_2$. Since every bounded set for the max metric is contained in a product of bounded sets, this family is cofinal in the bornology of $\rho_\times$-bounded subsets of $X_1 \times X_2$.

The following lemma records the distance formula for products of closed sets.

Lemma 6.4. *Let* $A \in CL(X_1)$ *and* $C \in CL(X_2)$*. Then for every* $(x_1, x_2) \in X_1 \times X_2$*,*

$$\rho \ \times \ \big((x_1, x_2), A \times C\big) = \max\big\{\rho_1(x_1, A), \ \rho_2(x_2, C)\big\}.$$

Proof. For $(a, c) \in A \times C$, we have

$$\rho \ \times \ \big((x_1, x_2), (a, c)\big) = \max\{\rho_1(x_1, a), \ \rho_2(x_2, c)\}.$$

Taking the infimum over $(a, c) \in A \times C$ gives

$$\rho \times \big((x_1, x_2), A \times C\big) = \inf_{a \in A,\ c \in C} \max\{\rho_1(x_1, a),\ \rho_2(x_2, c)\}.$$

Since the two variables separate, the infimum of the maximum is exactly the maximum of the two infima, hence

$$\rho \times \big((x_1, x_2), A \times C\big) = \max\left\{\inf_{a \in A} \rho_1(x_1, a),\ \inf_{c \in C} \rho_2(x_2, c)\right\} = \max\big\{\rho_1(x_1, A),\ \rho_2(x_2, C)\big\}.$$

□

We can now prove the product permanence theorem.

Theorem 6.5. *Let (A_n) and (A) be nonempty closed subsets of X_1, and let (C_n) and (C) be nonempty closed subsets of X_2. If $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_1}} A$ in X_1 and $C_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_2}} C$ in X_2, then*

$$A_n \times C_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_1 \boxtimes \mathcal{B}_2}} A \times C$$

in $X_1 \times X_2$.

Proof. Fix $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, and let $B := B_1 \times B_2$. Since ϕ is linearizing, Theorem 5.8 allows us to work directly with the linearized metrics. For $(x_1, x_2) \in B$, Lemma 6.4 gives

$$\rho \times \big((x_1, x_2), A_n \times C_n\big) = \max\{\rho_1(x_1, A_n),\ \rho_2(x_2, C_n)\}$$

and

$$\rho \times \big((x_1, x_2), A \times C\big) = \max\{\rho_1(x_1, A),\ \rho_2(x_2, C)\}.$$

Hence

$$\begin{aligned}\big|\rho \times \big((x_1, x_2), A_n \times C_n\big) - \rho_\times\big((x_1, x_2), A \times C\big)\big| \\ \le \max\Big\{\big|\rho_1(x_1, A_n) - \rho_1(x_1, A)\big|, \big|\rho_2(x_2, C_n) - \rho_2(x_2, C)\big|\Big\}.\end{aligned}$$

Taking the supremum over $(x_1, x_2) \in B$ yields

$$p_{\phi,B}(A_n \times C_n, A \times C) \le \max\big\{p_{\phi,B_1}(A_n, A),\ p_{\phi,B_2}(C_n, C)\big\}.$$

The right-hand side tends to zero by the assumed convergence in each factor. Since B_1 and B_2 were arbitrary, this proves the claim. □

By induction, the same argument extends to finite products of arbitrary length.

Remark 6.6. The proof of Theorem 6.5 extends directly to finite products. Hence, whenever a common linearizing scalarization is available, $(\phi, \mathcal{B})$-Attouch-Wets convergence is stable under finite products of closed sets.

6.3. Proper linearized geometry and relative compactness

We now assume that ϕ is linearizing and that the linearized metric

$$\rho(x, y) := \phi\big(d_\star(x, y)\big) \qquad (x, y \in X)$$

is proper. We also assume that $\mathcal{B}$ is the family of all ρ-bounded subsets of X. In this situation, $(\phi, \mathcal{B})$-Attouch-Wets convergence is exactly the bounded-uniform convergence of the distance functions $x \longmapsto \rho(x, A)$.

Fix a base point $x_0 \in X$, and for each $m \in \mathbb{N}$, set $K_m := \overline{B_\rho(x_0, m)}$. Since (X, ρ) is proper, each K_m is compact, and the family $\{K_m : m \in \mathbb{N}\}$ is an increasing cofinal subfamily in $\mathcal{B}$.

The next theorem gives a clean criterion for relative compactness in the hyperspace.

Theorem 6.7. *Assume that ϕ is linearizing, that (X, ρ) is proper, and that $\mathcal{B}$ is the family of all ρ-bounded subsets of X. Let $\mathcal{F} \subseteq CL(X)$. Then the following are equivalent:*

(1) every sequence in $\mathcal{F}$ has a subsequence that converges in the $(\phi, \mathcal{B})$-Attouch-Wets sense to a member of $CL(X)$;

(2) there exists $x_0 \in X$ such that $\sup_{A\in\mathcal{F}} \rho(x_0, A) < \infty$;

(3) for every $x \in X$, $\sup_{A\in\mathcal{F}} \rho(x, A) < \infty$.

Proof. We first show that (2) and (3) are equivalent. If (3) holds, then (2) is immediate. Now assume (2), and let $x \in X$ be arbitrary. For any $A \in \mathcal{F}$,

$$\rho(x, A) \le \rho(x, x_0) + \rho(x_0, A).$$

Taking the supremum over $A \in \mathcal{F}$ gives

$$\sup_{A\in\mathcal{F}} \rho(x, A) \le \rho(x, x_0) + \sup_{A\in\mathcal{F}} \rho(x_0, A) < \infty.$$

Thus (2)⇒(3).

It remains to show that (1)⇒(2). Suppose that (2) fails. Then for some fixed $x_0 \in X$, we can choose a sequence (A_n) in $\mathcal{F}$ such that $\rho(x_0, A_n) \to \infty$. If (A_{n_k}) were a convergent subsequence in $AW^\star_{\phi,\mathcal{B}}$ with limit $A \in CL(X)$, then by Theorem 5.10,

$$\rho(x_0, A_{n_k}) \to \rho(x_0, A) < \infty,$$

which is a contradiction. Hence (1) implies (2).

Finally, we prove (2)⇒(1). Let (A_n) be any sequence in $\mathcal{F}$, and assume that $\sup_n \rho(x_0, A_n) \le M$ for some $M > 0$.

For each n, choose $a_n \in A_n$ such that

$$\rho(x_0, a_n) \le \rho(x_0, A_n) + 1 \le M + 1.$$

Thus all a_n lie in the compact set K_{M+1}. Passing to a subsequence, we may assume that $a_n \to a \in K_{M+1}$.

For each n, define

$$f_n(x) := \rho(x, A_n) \qquad (x \in X).$$

Each f_n is 1-Lipschitz on (X, ρ). Indeed,

$$|f_n(x) - f_n(y)| = |\rho(x, A_n) - \rho(y, A_n)| \le \rho(x, y).$$

Moreover, for $x \in K_m$,

$$f_n(x) \le \rho(x, x_0) + \rho(x_0, A_n) \le m + M,$$

so the family $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded and equicontinuous on each compact set K_m.

By the Arzelà–Ascoli theorem, the sequence (f_n) has a subsequence converging uniformly on K_1. Applying the same argument successively on $K_2, K_3, \ldots$ and using a diagonal process, we obtain a subsequence, still denoted by (f_n), and a continuous function $f : X \to [0, \infty)$ such that $f_n \to f$ uniformly on each K_m. In particular, f is 1-Lipschitz.

We claim that the set $A := \{x \in X : f(x) = 0\}$ is nonempty and closed. Closedness follows from the continuity of f. To see that A is nonempty, note that $f_n(a_n) = \rho(a_n, A_n) = 0$. Since each f_n is 1-Lipschitz,

$$0 \le f_n(a) \le \rho(a, a_n) \longrightarrow 0.$$

Because $a \in K_{M+1}$ and $f_n \to f$ uniformly on K_{M+1}, it follows that $f(a) = 0$. Hence $a \in A$.

We now show that

$$f(x) = \rho(x, A) \qquad (x \in X).$$

Fix $x \in X$. Since f is 1-Lipschitz and $f = 0$ on A, for every $b \in A$,

$$f(x) \le f(b) + \rho(x, b) = \rho(x, b).$$

Taking the infimum over $b \in A$, gives $f(x) \le \rho(x, A)$.

For the reverse inequality, choose for each n a point $b_n \in A_n$ such that

$$\rho(x, b_n) \le \rho(x, A_n) + \frac{1}{n} = f_n(x) + \frac{1}{n}.$$

Since $f_n(x) \to f(x)$, the sequence $(f_n(x))$ is bounded, and therefore

$$\rho(x_0, b_n) \le \rho(x_0, x) + \rho(x, b_n)$$

is bounded as well. Thus (b_n) is contained in some compact set K_r. Passing to a subsequence, we may suppose that $b_n \to b \in K_r$. Now $f_n(b_n) = 0$ $(n \in \mathbb{N})$, and since each f_n is 1-Lipschitz,

$$0 \le f_n(b) \le \rho(b, b_n) \to 0.$$

Because $f_n \to f$ uniformly on K_r, we obtain $f(b) = 0$, so $b \in A$. On the other hand,

$$\rho(x, b) = \lim_{n\to\infty} \rho(x, b_n) = \lim_{n\to\infty} f_n(x) = f(x).$$

Hence $\rho(x, A) \le \rho(x, b) = f(x)$. Combining the two inequalities, we conclude that $f(x) = \rho(x, A)$ $(x \in X)$.

Since $f_n \to f$ uniformly on each K_m, and since

$$f_n(x) = \rho(x, A_n), \qquad f(x) = \rho(x, A),$$

it follows from Theorem 5.8 that $A_n \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} A$ along the extracted subsequence. This proves (1). □

The preceding theorem shows that, under properness of the linearized metric, relative compactness is characterized by a simple condition, namely, boundedness of the distance values at a single base point.

6.4. Sequential completeness of the scalarized hyperspace

The corresponding completeness statement is derived next.

Theorem 6.8. *Assume that ϕ is linearizing, that (X,ρ) is proper, and that $\mathcal{B}$ is the family of all ρ-bounded subsets of X. Then every $(\phi,\mathcal{B})$-Attouch-Wets Cauchy sequence in $CL(X)$ converges in $AW^{\star}_{\phi,\mathcal{B}}$ to a unique member of $CL(X)$.*

Equivalently, the pseudometric space $\big(CL(X),\delta_{\phi,\mathcal{B}}\big)$ is complete.

Proof. Let (A_n) be $(\phi,\mathcal{B})$-Attouch-Wets Cauchy. Since $\{x_0\}\in\mathcal{B}$ for every $x_0\in X$, Definition 4.9 yields

$$p_{\phi,\{x_0\}}(A_n,A_m)\to 0 \qquad (n,m\to\infty).$$

By Theorem 5.8,

$$p_{\phi,\{x_0\}}(A_n,A_m)=|\rho(x_0,A_n)-\rho(x_0,A_m)|.$$

Thus the sequence of real numbers $(\rho(x_0,A_n))$ is Cauchy, hence bounded. By Theorem 6.7, the family $\{A_n : n\in\mathbb{N}\}$ is relatively compact in $AW^{\star}_{\phi,\mathcal{B}}$. Therefore (A_n) has a subsequence (A_{n_k}) and a set $A\in CL(X)$ such that $A_{n_k}\xrightarrow{AW^{\star}_{\phi,\mathcal{B}}}A$.

We claim that the whole sequence converges to A. Let $B\in\mathcal{B}$ and $\varepsilon>0$ be given. Since (A_n) is Attouch-Wets Cauchy, there exists N_1 such that

$$p_{\phi,B}(A_n,A_m)<\frac{\varepsilon}{2} \qquad (n,m\geq N_1).$$

Since $A_{n_k}\to A$ in $AW^{\star}_{\phi,\mathcal{B}}$, there exists k_0 such that $n_{k_0}\geq N_1$ and $p_{\phi,B}(A_{n_{k_0}},A)<\frac{\varepsilon}{2}$. Then for every $n\geq N_1$, Proposition 3.3(3) gives

$$p_{\phi,B}(A_n,A)\leq p_{\phi,B}(A_n,A_{n_{k_0}})+p_{\phi,B}(A_{n_{k_0}},A)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence $p_{\phi,B}(A_n,A)\to 0$ $(n\to\infty)$. Since $B\in\mathcal{B}$ was arbitrary, we conclude that $A_n\xrightarrow{AW^{\star}_{\phi,\mathcal{B}}}A$.

Uniqueness follows from Theorem 4.8. The equivalent statement about the pseudometric $\delta_{\phi,\mathcal{B}}$ follows from Theorems 4.7 and 4.11. □

Remark 6.9. The results of this section show that $(\phi,\mathcal{B})$-Attouch-Wets convergence has the expected structural properties once a linearizing scalarization is fixed. Restriction to closed subspaces is automatic, the convergence is stable under finite products, and under properness of the linearized metric, the induced hyperspace admits both a relative compactness criterion and a sequential completeness theorem. In this sense, the present theory reproduces the main permanence and compactness features of the classical Attouch-Wets framework.

7. Examples and counterexamples

In this section, we present examples illustrating the nonlinear scope of the theory and the sharpness of its structural results. The comparison examples for the reverse implications in the bornological and Wijsman settings were given earlier in Section 5. The purpose of the present section is different. It

is to show that $(\phi, \mathcal{B})$-Attouch-Wets convergence remains meaningful in genuinely nonlinear $\star$-metric spaces, that the product result is not merely formal, and that the compactness criteria obtained in Section 6 are sharp. Accordingly, the examples below are devoted to the nonlinear model, the product theorem, and the sharpness of the compactness theory.

7.1. A genuine non-metric model

The nonlinear $\star$-metric from Example 2.6 will be used throughout this subsection. Recall that on $X = [0, \infty)$, we define

$$a \star b = (\sqrt{a} + \sqrt{b})^2, \qquad d_\star(x, y) = (\sqrt{x} - \sqrt{y})^2,$$

and we take the linearizing scalarization $\phi(t) = \sqrt{t}$. Then

$$\rho_\phi(x, y) = \phi(d_\star(x, y)) = |\sqrt{x} - \sqrt{y}|$$

is an ordinary metric on X, while $d_\star$ itself is not an ordinary metric.

Example 7.1. Let (t_n) be a sequence in $[0, \infty)$, let $t \in [0, \infty)$, and define

$$A_n = [t_n, \infty), \qquad A = [t, \infty).$$

Let $\mathcal{B}$ denote the family of all $d_\star$-bounded subsets of X, which coincides with the family of all ρ_ϕ-bounded subsets of X. Then $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$ if and only if (iff) $t_n \to t$.

Proof. For $x \in [0, \infty)$,

$$\rho_\phi(x, A_n) = \left(\sqrt{t_n} - \sqrt{x}\right)_+, \qquad \rho_\phi(x, A) = \left(\sqrt{t} - \sqrt{x}\right)_+.$$

Since the map $u \longmapsto (u - \sqrt{x})_+$ is 1-Lipschitz on $[0, \infty)$ for each fixed x, we obtain

$$\left|\rho_\phi(x, A_n) - \rho_\phi(x, A)\right| \le |\sqrt{t_n} - \sqrt{t}|.$$

Therefore, for every $B \in \mathcal{B}$,

$$p_{\phi,B}(A_n, A) = \sup_{x \in B} \left|\rho_\phi(x, A_n) - \rho_\phi(x, A)\right| \le |\sqrt{t_n} - \sqrt{t}|.$$

Hence $t_n \to t$ implies $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.

Conversely, if $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$, then in particular, since $\{0\} \in \mathcal{B}$,

$$p_{\phi,\{0\}}(A_n, A) \to 0.$$

But

$$p_{\phi,\{0\}}(A_n, A) = \left|\rho_\phi(0, A_n) - \rho_\phi(0, A)\right| = |\sqrt{t_n} - \sqrt{t}|.$$

Thus $\sqrt{t_n} \to \sqrt{t}$, and therefore $t_n \to t$. □

This example is representative of the nonlinear setting considered in the paper. It shows that $(\phi, \mathcal{B})$-Attouch-Wets convergence remains meaningful in a genuinely non-metric $\star$-space and that, under a linearizing scalarization, it can be described through the associated distance profiles. A schematic illustration of the half-lines $A_n = [t_n, \infty)$ and of the limit set $A = [t, \infty)$ is given in Figure 1. The figure serves only to visualize the underlying set geometry; the convergence itself is determined by the bounded-region uniform behavior of the profiles $x \mapsto \rho_\phi(x, A_n)$ and $x \mapsto \rho_\phi(x, A)$, which characterizes the relation $A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}}} A$.

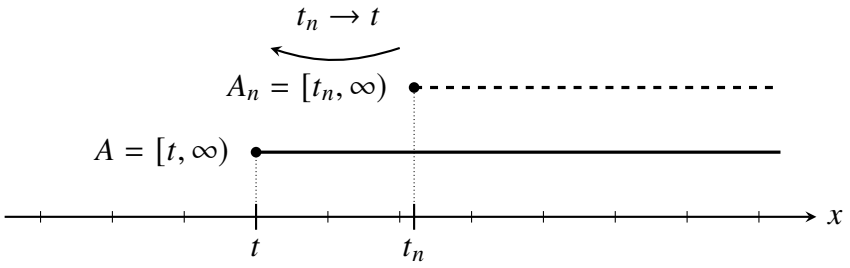


Figure 1. The half-lines $A_n = [t_n, \infty)$ and $A = [t, \infty)$ from Example 7.1 are shown. The picture illustrates the underlying set geometry, while the convergence is determined by the scalarized profiles on bounded testing regions.

7.2. A product example in the nonlinear model

The next example illustrates the product theorem in a genuinely nonlinear $\star$-metric setting.

Example 7.2. For $i = 1, 2$, let

$$X_i = [0, \infty), \qquad a \star b = (\sqrt{a} + \sqrt{b})^2, \qquad d_{\star,i}(x, y) = (\sqrt{x} - \sqrt{y})^2,$$

and take $\phi(t) = \sqrt{t}$. Let (s_n) and (t_n) be sequences in $[0, \infty)$ with $s_n \to s$ and $t_n \to t$. Define

$$A_n = [s_n, \infty) \subseteq X_1, \qquad C_n = [t_n, \infty) \subseteq X_2,$$

and

$$A = [s, \infty), \qquad C = [t, \infty).$$

Then

$$A_n \times C_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_1 \boxtimes \mathcal{B}_2}} A \times C,$$

where each $\mathcal{B}_i$ is the family of all bounded subsets of X_i.

Proof. By Example 7.1,

$$A_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_1}} A \qquad \text{and} \qquad C_n \xrightarrow{AW^\star_{\phi,\mathcal{B}_2}} C.$$

Hence the conclusion follows at once from Theorem 6.5. □

This example shows that the product theorem is not merely a formal counterpart of the classical metric result, but remains valid in an ambient setting where the original $\star$-distance is genuinely nonlinear.

7.3. Sharpness of the relative compactness criterion

We close the section with two examples illustrating the criterion from Theorem 6.7.

Example 7.3. Let

$$X = \mathbb{R}, \qquad a \star b = a + b, \qquad \phi(t) = t,$$

and let $\mathcal{B}$ be the family of all bounded subsets of $\mathbb{R}$. For a fixed $M > 0$, consider the family

$$\mathcal{F}_M = \{[r, \infty) : 0 \le r \le M\} \subseteq CL(\mathbb{R}).$$

Then every sequence in $\mathcal{F}_M$ has a subsequence that converges in the $(\phi, \mathcal{B})$-Attouch-Wets sense to some member of $CL(\mathbb{R})$.

Proof. Fix the base point $x_0 = 0$. For every $A = [r, \infty) \in \mathcal{F}_M$,

$$d(0, A) = r \le M.$$

Hence $\sup_{A \in \mathcal{F}_M} d(0, A) \le M$. Since $(\mathbb{R}, d)$ is proper, Theorem 6.7 yields the desired relative compactness. One may also see this directly: from any sequence

$$A_n = [r_n, \infty) \qquad (0 \le r_n \le M),$$

one can extract a subsequence with $r_{n_k} \to r \in [0, M]$, and the same elementary distance-profile computation as in Example 7.1 shows that

$$A_{n_k} \xrightarrow{AW^{\star}_{\phi,\mathcal{B}}} [r, \infty).$$

□

Example 7.4. Let

$$X = \mathbb{R}, \qquad a \star b = a + b, \qquad \phi(t) = t,$$

and let $\mathcal{B}$ be the family of all bounded subsets of $\mathbb{R}$. Consider the family

$$\mathcal{G} = \{[n, \infty) : n \in \mathbb{N}\} \subseteq CL(\mathbb{R}).$$

Then $\mathcal{G}$ is not relatively compact with respect to $(\phi, \mathcal{B})$-Attouch-Wets convergence.

Proof. Take $x_0 = 0$. For $A_n = [n, \infty)$, we have $d(0, A_n) = n \to \infty$. Therefore $\sup_{A \in \mathcal{G}} d(0, A) = \infty$. By Theorem 6.7, the family $\mathcal{G}$ is not relatively compact with respect to $(\phi, \mathcal{B})$-Attouch-Wets convergence.

One can also argue directly. If a subsequence (A_{n_k}) were $(\phi, \mathcal{B})$-Attouch-Wets convergent to some $A \in CL(\mathbb{R})$, then Theorem 5.10 would give

$$d(0, A_{n_k}) \to d(0, A) < \infty.$$

This is impossible, since $d(0, A_{n_k}) = n_k \to \infty$. □

Examples 7.3 and 7.4 show that the base-point boundedness condition in Theorem 6.7 is essential. It marks the boundary between relative compactness in the hyperspace and escape to infinity.

Remark 7.5. The examples in this section illustrate the nonlinear scope and structural sharpness of the present theory. Example 7.1 shows that $(\phi, \mathcal{B})$-Attouch-Wets convergence remains meaningful in a genuinely nonlinear $\star$-metric space. Example 7.2 shows that the product theorem is not merely formal in the nonlinear setting. Finally, Examples 7.3 and 7.4 show that the compactness criterion of Theorem 6.7 is sharp.

8. Conclusions and further directions

In this paper, we developed an Attouch-Wets-type convergence theory for closed sets in $\star$-metric spaces by combining bounded-region hyperspace control with a scalarization of the underlying $\star$-distance. The central point of the construction is that the nonlinear triangle structure of a $\star$-metric space does not allow a direct transfer of the classical bounded-Hausdorff framework, whereas an admissible scalarization makes it possible to recover an additive profile geometry on bounded testing regions. This yields a natural route from the nonlinear ambient distance to a coherent hyperspace convergence theory.

The first part of the paper established the basic functionals underlying this setting. Profile functionals and truncated excess functionals associated with an admissible scalarization were introduced, their finiteness on bounded testing regions was proved, and the monotonicity, symmetry, and triangle-type estimates needed for a workable convergence scheme were established. This led to the definition of $(\phi, \mathcal{B})$-Attouch-Wets convergence as the bounded-region uniform convergence of the corresponding distance profiles.

The second part of the paper clarified the internal structure of this convergence. Equivalent one-sided formulations were obtained, it was shown that any cofinal subfamily of the testing family determines the convergence, and pseudometrizability was established whenever the testing family has a countable cofinal subfamily. In the same setting, the associated Cauchy theory is naturally described by the induced hyperspace pseudometric, and the convergence has unique limits. Thus $(\phi, \mathcal{B})$-Attouch-Wets convergence is not merely a formal analogue of a known metric construction, but a coherent hyperspace structure in its own right.

A central theme of the paper was the comparison with other convergence modes. It was shown that $(\phi, \mathcal{B})$-Attouch-Wets convergence always implies the corresponding upper, lower, and two-sided bornological convergences induced by the same testing family. It was also shown that it implies Wijsman convergence, and that this implication is strict in general. On the other hand, when the scalarization is linearizing and the associated linearized metric is proper, the pointwise and bounded-uniform theories coincide. In this way, the present theory recovers the expected Attouch-Wets behavior in the appropriate linearized setting while remaining meaningful in genuinely non-metric $\star$-spaces.

The structural results of the paper show that this convergence behaves as a genuine hyperspace convergence should. Restriction to closed subspaces is automatic, the convergence is stable under finite products whenever a common linearizing scalarization is available, and under properness of the linearized metric, the associated hyperspace admits both a relative compactness criterion and a sequential completeness theorem. These results show that $(\phi, \mathcal{B})$-Attouch-Wets convergence is not only definable in the $\star$-metric setting, but also stable under the basic operations of hyperspace theory and strong enough to support a useful compactness theory.

The examples and counterexamples demonstrate that the theory captures genuinely new phenomena. In particular, they show that one-sided bornological control is not sufficient to recover bounded-profile convergence, that Wijsman convergence need not upgrade to $(\phi, \mathcal{B})$-Attouch-Wets convergence in the absence of properness, and that the theory remains meaningful in nonlinear $\star$-metric models that do not reduce to the ordinary metric case. These features also support the application-oriented relevance of the framework, which provides a natural language for studying moving feasible sets in nonlinear optimization, robustness analysis, and other generalized metric-type models in which the ambient

dissimilarity is not additive.

Several directions for further study remain open. One may investigate finer criteria for total boundedness and compactness in the induced hyperspace, especially beyond the proper case. It would also be natural to study more systematically the dependence of the theory on the choice of scalarization and to identify classes of scalarizations that generate equivalent hyperspace structures. Another promising direction is to extend the present setting to ideal, deferred, lacunary, or modular variants, where bounded-region profile control is combined with generalized asymptotic methods. Finally, the present approach may provide a useful language for evolving feasible-set models in nonlinear optimization, variational analysis, and related problems in which the ambient geometry is not naturally additive.

Taken together, the results of this paper place hyperspace convergence in $\star$-metric spaces on a firmer foundation. They show that, after passing through a suitable scalarization, the classical bounded-region theory of closed-set convergence can be extended to the nonlinear $\star$-setting without losing either geometric content or structural depth. This point of view may provide a basis for further developments in both generalized metric geometry and asymptotic set analysis.

Author contributions

Conceptualization, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; methodology, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; validation, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; formal analysis, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; investigation, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; writing—original draft preparation, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; writing—review and editing, Q. W., M. G., S. S., S. Ç., O. Y., and Q. B. C.; supervision, M. G. and Q. B. C. All authors contributed equally to this work and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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