文章编号:1674-2974(2024)01-0101-11 DOI:10.16339/j.cnki.hdxbzkb.2024010

四边不同支承条件下矩形板的结构计算

杨成永[1],许清滔[2],马文辉[3†],韩薛果[1]
(1. 北京交通大学 土木建筑工程学院,北京 100044;
2. 安徽省交通规划设计研究总院股份有限公司,安徽 合肥 230088;
3. 北京城建轨道交通建设工程有限公司,北京 100176)

摘 要:采用带补充项的傅立叶级数作为挠度函数,针对四边不同支承矩形薄板,推导了确定待定系数的方程组,给出可处理简支边、固支边和自由边任意组合条件下统一的结构计算公式.探讨了集中荷载作用处弯矩级数解不收敛的处理办法,以及双向板简化为单向板需要达到的长宽比问题.结果表明,集中荷载作用处的弯矩,可采用挠度值按中心差分公式进行计算,差分步长可取 10 mm.对边支承对边自由板及一边固支三边自由板,可视作单向板.当四边支承板的长宽比达到 2∶1、2.5∶1 及 4.5∶1 时,可分别简化为两端固支、一端简支一端固支及两端简支单向板.三边支承一边自由板长宽比达到 1∶1 及 2∶1 时,可分别简化为两端固支(及一端简支一端固支)及两端简支单向板;长宽比达到 6∶1 时,可简化为悬臂单向板.两邻边支承两邻边自由板若要简化为悬臂单向板,在两支承边为固支时,长宽比需要达到 2∶1;在支承边为一边简支一边固支时,长宽比要达到 1.5∶1.

关键词:矩形薄板;结构分析;支承条件;级数解;收敛;单向板
中图分类号:U411 **文献标志码**:A

Structural Calculation of Rectangular Plates with Different Support Conditions of Four Edges

YANG Chengyong[1], XU Qingtao[2], MA Wenhui[3†], HAN Xueguo[1]
(1. School of Civil Engineering, Beijing Jiaotong University, Beijing 100044, China;
2. Anhui Transport Consulting & Design Research Institute Co., Ltd., Hefei 230088, China;
3. Beijing Urban Rail Transit Construction Engineering Co., Ltd., Beijing 100176, China)

Abstract: For a rectangular thin plate with different support conditions of four edges, a double Fourier series with additional terms was taken as the deflection function, and linear algebraic equations for solving the undetermined coefficients were derived. A unified structural calculation formula was obtained for the plate with any combination of simply supported, clamped, and free edges. The non-convergence of a series solution of bending moment at a point where a concentrated load is applied was discussed. In addition, the aspect ratio problem that needs to be achieved when simplifying a two-way plate is also discussed. The results show that the bending moment at the point can be calculated through the finite difference method using the deflection value, where a differential

* **收稿日期**:2023-02-08
基金项目:国家自然科学基金高铁联合基金重点支持项目(U1934210),Key Project by High Speed Railway Joint Fund, National Natural Science Foundation of China (U1934210)
作者简介:杨成永(1966—),男,贵州遵义人,北京交通大学教授,博士
† 通信联系人,E-mail: mwh.bjtu@foxmail.com

step length of 10 mm was selected. A plate with two opposite edges supported and two other edges free, and a plate with one edge clamped and the other three edges free, can be taken as a one-way plate. A plate with all four edges supported can be treated as a one-way plate with two ends clamped, one end simply supported and the other end clamped, or two ends simply supported if its aspect ratio reaches 2∶1, 2.5∶1, or 4.5∶1, respectively. A plate with three edges supported and the last edge free can be seen as a one-way plate with two ends clamped (and one end simply supported and the other end clamped), or two ends simply supported if its aspect ratio gets 1∶1 or 2∶1, separately; this type of plate can also be simplified as a cantilevered one-way plate at an aspect ratio of no less than 6∶1. A plate with two adjacent edges supported and the other two edges free can be taken as a cantilevered one-way plate if its aspect ratio reaches 2∶1 when the two supported edges are all clamped, or if the aspect ratio reaches1.5∶1 when one edge is clamped and an adjacent edge is simply supported.

Key words: rectangular thin plate; structural analysis; support conditions; series solution; convergence; one-way plate

矩形薄板的结构计算可采用叠加法[1]、改进傅立叶级数法[2]、辛几何法[3]、积分变换法[4]等方法进行处理. 例如:许琪楼等[5]采用叠加原理,给出了四边支承矩形板的统一求解方法. 杨端生等[6]提出了板弯曲变形问题的一般解析法,即采用双正弦级数、单正弦级数、代数多项式作为挠度函数,根据四个边和四个角的边界条件确定待定系数. 杨成永等[7]采用改进傅立叶级数法研究了局部均布荷载作用下四边支承矩形板的弯曲变形问题. 钟阳等[8]利用辛几何法导出了四边固支矩形板的解析表达式. 张景辉[9]通过积分变换解法求解了两邻边自由另两边固支或简支薄板,并将傅立叶积分变换法与叠加原理相结合,得到了点支承边界条件下的解析解. 陈英杰等[10]采用混合能量原理得到了静水压力作用下三边简支一边固定、两邻边固定两邻边简支、三边固定一边简支矩形板的挠曲面方程. 李状飞等[11]讨论了在线荷载作用下三边固支一边自由矩形板的弯曲变形问题,以线荷载为界把板分成两个区域,分区域采用相同形式的挠度函数.

但以往的研究成果一般只是针对具体的一种或几种边界条件给出了问题的解,没有提供板边在简支、固支和自由任意组合条件下的计算公式,应用上有所不便. 此外,在集中荷载作用处,存在弯矩计算不收敛的问题,需要进行处理. 再者,对于单向板与双向板区分界限的问题,目前的研究结果仅限于四边支承板[12-13],且在研究方法上,由于缺乏任意边界组合情况下的解析计算公式,主要是采用有限元数值方法.

本文采用改进傅立叶级数法,给出了板边简支、固支和自由任意组合情况下待定系数的统一计算公式;根据挠度的计算值采用差分方法计算了集中荷载作用点处的弯矩,确定差分步长的取值;最后利用本文公式通过计算提出不同支承条件下双向板简化为单向板需要达到的长宽比.

1 板弯曲变形的基本方程

矩形板承受荷载的类型和参数如图1所示. 坐标系设定为以板左下角点为原点,x轴向右为正,y轴向上为正.

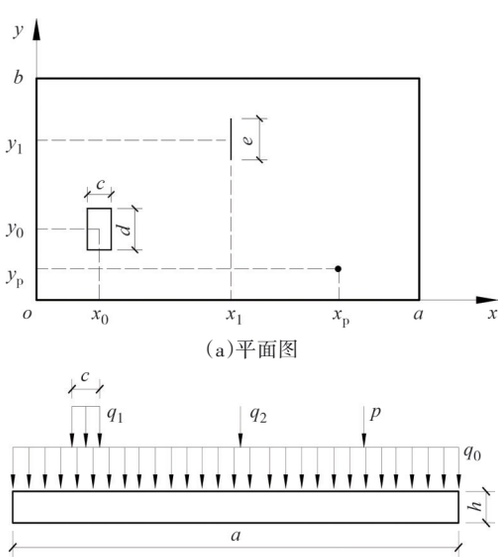


图1 受荷载作用的矩形板

Fig.1 A rectangular plate under loads

图中:a、b为板的长度和宽度,m;h为板的厚度,m;q_0为满布均布荷载,kPa;q_1为局部均布荷载,kPa;c、d分别为局部均布荷载的分布长度和宽度,m;x_0、y_0为局部均布荷载中心的坐标,m;q_2为线荷载,kN/m;e为线荷载沿y方向的长度,m;x_1、y_1为线荷载中心的坐标,m;p为集中荷载,kN;x_p、y_p为集中荷载作用点的坐标,m.

板弯曲变形的基本微分方程为[1]:

$$D\nabla^2\nabla^2 w(x,y)=q(x,y) \tag{1}$$

式中:∇^2为拉普拉斯算子,$\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$;$D$为板的抗弯刚度,$D=\dfrac{Eh^3}{12(1-v^2)}$;$E$为板的弹性模量,kPa;$v$为板的泊松比.

公式(1)中右端荷载$q(x,y)$的傅立叶级数展开为[14]:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2}$$

对于满布均布荷载,公式(2)中的傅立叶系数q_{mn}为:

$$q_{mn}=\frac{16q_0}{\pi^2 mn}\qquad m,n=1,3,5,\cdots \tag{3}$$

对局部均布荷载:

$$q_{mn}=\frac{16q_1}{\pi^2 mn}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}\sin\frac{m\pi c}{2a}\sin\frac{n\pi d}{2b}$$
$$m,n=1,2,3,\cdots \tag{4}$$

对于沿y方向的线荷载:

$$q_{mn}=\frac{8q_2}{\pi an}\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{n\pi e}{2b}$$
$$m,n=1,2,3,\cdots \tag{5}$$

对于集中荷载:

$$q_{mn}=\frac{4p}{ab}\sin\frac{m\pi x_p}{a}\sin\frac{n\pi y_p}{b}$$
$$m,n=1,2,3,\cdots \tag{6}$$

2　挠度、弯矩和剪力计算

对板边为简支、固支和自由三种不同支承条件下的矩形板,有带补充项的挠度表达式为[2]

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}+\sum_{n=1}^{\infty}f_n(x)\sin\frac{n\pi y}{b}+\sum_{m=1}^{\infty}f_m(y)\sin\frac{m\pi x}{a}+$$
$$\frac{xy}{ab}\left(w_{00}+w_{ab}-w_{a0}-w_{0b}\right)+\frac{x}{a}\left(w_{a0}-w_{00}\right)+\frac{y}{b}\left(w_{0b}-w_{00}\right)+w_{00} \tag{7}$$

式中:w_{00}、w_{a0}、w_{0b}、w_{ab}分别为矩形板在邻边自由时左下、右下、左上、右上角点的沉降,m.

$$f_n(x)=\frac{1}{6a}\left[\frac{E_n-F_n}{D}+\pi^2\beta_n^2 v\left(B_n-A_n\right)\right]x^3+\frac{1}{2}\left(\pi^2\beta_n^2 vA_n-\frac{E_n}{D}\right)x^2+$$
$$\left[\frac{B_n-A_n}{a}+\frac{a}{6D}\left(F_n+2E_n\right)-\frac{\pi^2\beta_n^2 av\left(B_n+2A_n\right)}{6}\right]x+A_n \tag{8}$$

$$f_m(y)=\frac{1}{6b}\left[\frac{G_m-H_m}{D}+\pi^2\alpha_m^2 v\left(D_m-C_m\right)\right]y^3+\frac{1}{2}\left(\pi^2\alpha_m^2 vC_m-\frac{G_m}{D}\right)y^2+$$
$$\left[\frac{D_m-C_m}{b}+\frac{b}{6D}\left(H_m+2G_m\right)-\frac{\pi^2\alpha_m^2 bv\left(D_m+2C_m\right)}{6}\right]y+C_m \tag{9}$$

式中:$\alpha_m=\dfrac{m}{a}$;$\beta_n=\dfrac{n}{b}$.

公式(7)中w_{mn}为挠度的傅立叶系数:

$$w_{mn}=\frac{1}{D\pi^4}\frac{1}{\left(\alpha_m^2+\beta_n^2\right)^2}\left\{-\frac{2\pi\beta_n^2}{m}\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)\left[E_n+(-1)^{m+1}F_n\right]-\frac{2\pi\alpha_m^2}{n}\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)\left[G_m+(-1)^{n+1}H_m\right]+\right.$$
$$\left.\frac{2\pi^3\beta_n^4 D}{m}\left[v\left(\frac{\beta_n^2}{\alpha_m^2}+2\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]+\frac{2\pi^3\alpha_m^4 D}{n}\left[v\left(\frac{\alpha_m^2}{\beta_n^2}+2\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]+q_{mn}\right\} \tag{10}$$

公式(8)~(10)中的A_n、B_n、C_m、D_m分别为板左边(x=0边)、右边(x=a边)、下边(y=0边)、上边(y=b边)处挠度正弦级数的待定系数;相应地,E_n、F_n、G_m、H_m分别为板左边、右边、下边和上边处法向弯矩正弦级

数的待定系数.

板四边上挠度正弦级数表达式为:

$$\begin{cases} w(0,y)=\sum\limits_{n=1}^{\infty}A_n\sin\dfrac{n\pi y}{b}+w_{0b}\dfrac{y}{b}+w_{00}\left(1-\dfrac{y}{b}\right) \\ w(a,y)=\sum\limits_{n=1}^{\infty}B_n\sin\dfrac{n\pi y}{b}+w_{ab}\dfrac{y}{b}+w_{a0}\left(1-\dfrac{y}{b}\right) \\ w(x,0)=\sum\limits_{m=1}^{\infty}C_m\sin\dfrac{m\pi x}{a}+w_{a0}\dfrac{x}{a}+w_{00}\left(1-\dfrac{x}{a}\right) \\ w(x,b)=\sum\limits_{m=1}^{\infty}D_m\sin\dfrac{m\pi x}{a}+w_{ab}\dfrac{x}{a}+w_{0b}\left(1-\dfrac{x}{a}\right) \end{cases} \tag{11}$$

弯矩正弦级数表达式为:

$$\begin{cases} M_x(0,y)=\sum\limits_{n=1}^{\infty}E_n\sin\dfrac{n\pi y}{b} \\ M_x(a,y)=\sum\limits_{n=1}^{\infty}F_n\sin\dfrac{n\pi y}{b} \\ M_y(x,0)=\sum\limits_{m=1}^{\infty}G_m\sin\dfrac{m\pi x}{a} \\ M_y(x,b)=\sum\limits_{m=1}^{\infty}H_m\sin\dfrac{m\pi x}{a} \end{cases} \tag{12}$$

利用公式(7)~(9),有任意点处x方向的弯矩为:

$$\begin{aligned} M_x=&-D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)=D\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\alpha_m^2+v\beta_n^2)w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}- \\ &\sum_{n=1}^{\infty}\left\{\frac{1}{a}\left[E_n-F_n+\pi^2Dv\beta_n^2(B_n-A_n)\right]x+\pi^2Dv\beta_n^2A_n-E_n-\pi^2Dv\beta_n^2f_n(x)\right\}\sin\frac{n\pi y}{b}+ \\ &\sum_{m=1}^{\infty}\left(\pi^2D\alpha_m^2f_m(y)-v\left\{\frac{1}{b}\left[G_m-H_m+\pi^2Dv\alpha_m^2(D_m-C_m)y+\pi^2Dv\alpha_m^2C_m-G_m\right]\right\}\right)\sin\frac{m\pi x}{a} \end{aligned} \tag{13}$$

y方向的弯矩为:

$$\begin{aligned} M_y=&-D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right)=D\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\beta_n^2+v\alpha_m^2)w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}- \\ &\sum_{m=1}^{\infty}\left\{\frac{1}{b}\left[G_m-H_m+\pi^2Dv\alpha_m^2(D_m-C_m)\right]y+\pi^2Dv\alpha_m^2C_m-G_m-\pi^2Dv\alpha_m^2f_m(y)\right\}\sin\frac{m\pi x}{a}+ \\ &\sum_{n=1}^{\infty}\left(\pi^2D\beta_n^2f_n(x)-v\left\{\frac{1}{a}\left[E_n-F_n+\pi^2Dv\beta_n^2(B_n-A_n)\right]x+\pi^2Dv\beta_n^2A_n-E_n\right\}\right)\sin\frac{n\pi y}{b} \end{aligned} \tag{14}$$

在任意点处x方向的剪力为:

$$\begin{aligned} V_x=&-D\left[\frac{\partial^3 w}{\partial x^3}+(2-v)\frac{\partial^3 w}{\partial x\partial y^2}\right]=D\pi^3\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]w_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}- \\ &\sum_{n=1}^{\infty}\left\{\frac{1}{a}\left[E_n-F_n+\pi^2Dv\beta_n^2(B_n-A_n)\right]-\pi^2D(2-v)\beta_n^2g_n(x)\right\}\sin\frac{n\pi y}{b}+ \\ &\sum_{m=1}^{\infty}\left\{\pi^2D\alpha_m^2f_m(y)-(2-v)\left(\frac{1}{b}\left[G_m-H_m+\pi^2Dv\alpha_m^2(D_m-C_m)\right]y+\pi^2Dv\alpha_m^2C_m-G_m\right)\right\}\alpha_m\pi\cos\frac{m\pi x}{a} \end{aligned} \tag{15}$$

y方向的剪力为:

$$\begin{aligned} V_y=&-D\left[\frac{\partial^3 w}{\partial y^3}+(2-v)\frac{\partial^3 w}{\partial y\partial x^2}\right]=D\pi^3\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]w_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}- \\ &\sum_{m=1}^{\infty}\left\{\frac{1}{b}\left[G_m-H_m+\pi^2Dv\alpha_m^2(D_m-C_m)\right]-\pi^2D(2-v)\alpha_m^2g_m(y)\right\}\sin\frac{m\pi x}{a}+ \\ &\sum_{n=1}^{\infty}\left\{\pi^2D\beta_n^2f_n(x)-(2-v)\left(\frac{1}{a}\left[E_n-F_n+\pi^2Dv\beta_n^2(B_n-A_n)\right]x+\pi^2Dv\beta_n^2A_n-E_n\right)\right\}\beta_n\pi\cos\frac{n\pi y}{b} \end{aligned} \tag{16}$$

式中:

$$g_n(x)=\frac{\mathrm{d}f_n(x)}{\mathrm{d}x}=\frac{1}{2a}\left[\frac{E_n-F_n}{D}+\pi^2v\beta_n^2\left(B_n-A_n\right)\right]x^2+\left(\pi^2v\beta_n^2A_n-\frac{E_n}{D}\right)x+$$

$$\left[\frac{B_n-A_n}{a}+\frac{a\left(F_n+2E_n\right)}{6D}-\frac{\pi^2v\beta_n^2a\left(B_n+2A_n\right)}{6}\right] \tag{17}$$

$$g_m(y)=\frac{\mathrm{d}f_m(y)}{\mathrm{d}y}=\frac{1}{2b}\left[\frac{G_m-H_m}{D}+\pi^2v\alpha_m^2\left(D_m-C_m\right)\right]y^2+\left(\pi^2v\alpha_m^2C_m-\frac{G_m}{D}\right)y+$$

$$\left[\frac{D_m-C_m}{b}+\frac{b\left(H_m+2G_m\right)}{6D}-\frac{\pi^2v\alpha_m{}^2b\left(D_m+2C_m\right)}{6}\right] \tag{18}$$

3　待定系数的确定

公式(8)~(18)中待定系数A_n、B_n、C_m、D_m及E_n、F_n、G_m、H_m需要根据板边的支承条件确定.对简支或固支边,由公式(11)知,挠度待定系数A_n、B_n、C_m、D_m为0.对简支或自由边,由公式(12)知,弯矩待定系数E_n、F_n、G_m、H_m为0.

当板有固支边时,根据左边、右边、下边及上边的边界条件$\frac{\partial w}{\partial x}\big|_{x=0}=0$、$\frac{\partial w}{\partial x}\big|_{x=a}=0$、$\frac{\partial w}{\partial y}\big|_{y=0}=0$、$\frac{\partial w}{\partial y}\big|_{y=b}=0$,利用公式(7)~(10),并在推导过程中把出现的常数项及1~3次方幂函数项展开成正弦级数,可以得到确定待定系数E_n、F_n、G_m、H_m的方程分别为:

$$\frac{B_n}{a}+\frac{a}{6D}\left(F_n+2E_n\right)-\frac{\pi^2v\beta_n^2aB_n}{6}-\frac{2\beta_n^2}{D\pi^2a}\sum_{m=1}^{\infty}\frac{1}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)\left[E_n+(-1)^{m+1}F_n\right]+$$

$$\frac{2n}{D\pi^2ab^2}\sum_{m=1}^{\infty}\frac{m}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[G_m+(-1)^{n+1}H_m\right]+\frac{2\beta_n^4}{a}\sum_{m=1}^{\infty}\frac{1}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(\frac{\beta_n^2}{\alpha_m^2}+2\right)-1\right](-1)^{m+1}B_n+$$

$$\frac{2}{n}\sum_{m=1}^{\infty}\frac{\alpha_m^5}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]-\frac{2vb^2}{an^3}\sum_{m=1}^{\infty}\alpha_m^2\left[C_m+(-1)^{n+1}D_m\right]m+$$

$$\frac{2}{na}\sum_{m=1}^{\infty}\left[C_m+(-1)^{n+1}D_m\right]m+\frac{1}{D\pi^3}\sum_{m=1}^{\infty}\frac{a_m}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+\frac{2(-1)^{n+1}}{a\pi n}\left(w_{ab}-w_{0b}\right)+\frac{2}{a\pi n}\left(w_{a0}-w_{00}\right)=0 \tag{19}$$

$$-\frac{A_n}{a}-\frac{a}{6D}\left(2F_n+E_n\right)+\frac{\pi^2v\beta_n^2aA_n}{6}-\frac{2\beta_n^2}{D\pi^2a}\sum_{m=1}^{\infty}\frac{(-1)^m}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)\left[E_n+(-1)^{m+1}F_n\right]+$$

$$\frac{2n}{D\pi^2ab^2}\sum_{m=1}^{\infty}\frac{(-1)^m m}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[G_m+(-1)^{n+1}H_m\right]+\frac{2\beta_n^4}{a}\sum_{m=1}^{\infty}\frac{(-1)^m}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(\frac{\beta_n^2}{\alpha_m^2}+2\right)-1\right]A_n+$$

$$\frac{2}{n}\sum_{m=1}^{\infty}\frac{(-1)^m\alpha_m^5}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]-\frac{2vb^2}{an^3}\sum_{m=1}^{\infty}\alpha_m^2\left[C_m+(-1)^{n+1}D_m\right]m(-1)^m+$$

$$\frac{2}{na}\sum_{m=1}^{\infty}\left[C_m+(-1)^{n+1}D_m\right]m(-1)^m+\frac{1}{D\pi^3}\sum_{m=1}^{\infty}\frac{a_m(-1)^m}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+$$

$$\frac{2(-1)^{n+1}}{a\pi n}\left(w_{ab}-w_{0b}\right)+\frac{2}{a\pi n}\left(w_{a0}-w_{00}\right)=0 \tag{20}$$

$$\frac{D_m}{b}+\frac{b}{6D}\left(H_m+2G_m\right)-\frac{\pi^2 v\alpha_m^2 bD_m}{6}-\frac{2\alpha_m^2}{D\pi^2 b}\sum_{n=1}^{\infty}\frac{1}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)\left[G_m+(-1)^{n+1}H_m\right]+$$

$$\frac{2m}{D\pi^2 a^2 b}\sum_{n=1}^{\infty}\frac{n}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[E_n+(-1)^{m+1}F_n\right]+\frac{2\alpha_m^4}{b}\sum_{n=1}^{\infty}\frac{1}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(\frac{\alpha_m^2}{\beta_n^2}+2\right)-1\right](-1)^{n+1}D_m+$$

$$\frac{2}{m}\sum_{n=1}^{\infty}\frac{\beta_n^5}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]-\frac{2va^2}{bm^3}\sum_{n=1}^{\infty}\beta_n^2\left[A_n+(-1)^{m+1}B_n\right]n+$$

$$\frac{2}{mb}\sum_{n=1}^{\infty}\left[A_n+(-1)^{m+1}B_n\right]n+\frac{1}{D\pi^3}\sum_{n=1}^{\infty}\frac{\beta_n}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+\frac{2(-1)^{m+1}}{b\pi m}\left(w_{ab}-w_{a0}\right)+\frac{2}{b\pi m}\left(w_{0b}-w_{00}\right)=0 \tag{21}$$

$$-\frac{C_n}{b}-\frac{b}{6D}\left(2H_m+G_m\right)+\frac{\pi^2 v\alpha_m^2 bC_n}{6}-\frac{2\alpha_m^2}{D\pi^2 b}\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)\left[G_m+(-1)^{n+1}H_m\right]+$$

$$\frac{2m}{D\pi^2 a^2 b}\sum_{n=1}^{\infty}\frac{(-1)^n n}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[E_n+(-1)^{m+1}F_n\right]+\frac{2\alpha_m^4}{b}\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(\frac{\alpha_m^2}{\beta_n^{\,2}}+2\right)-1\right]C_m+$$

$$\frac{2}{m}\sum_{n=1}^{\infty}\frac{(-1)^n\beta_n^5}{\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]-\frac{2va^2}{bm^3}\sum_{n=1}^{\infty}\beta_n^2\left[A_n+(-1)^{m+1}B_n\right]n(-1)^n+$$

$$\frac{2}{mb}\sum_{n=1}^{\infty}\left[A_n+(-1)^{m+1}B_n\right]n(-1)^n+\frac{1}{D\pi^3}\sum_{m=1}^{\infty}\frac{\beta_n(-1)^n}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+\frac{2(-1)^{m+1}}{b\pi m}\left(w_{ab}-w_{a0}\right)+\frac{2}{b\pi m}\left(w_{0b}-w_{00}\right)=0 \tag{22}$$

当板有自由边时,根据左边、右边、下边及上边的边界条件件$V_x\big|_{x=0}=0$、$V_x\big|_{x=a}=0$、$V_y\big|_{y=0}=0$、$V_y\big|_{y=b}=0$,考虑公式(15)~(18),利用公式(8)~(10),并在推导过程中把出现的常数项及1~3次方幂函数项展开成正弦级数,可以得到确定待定系数A_n、B_n、C_m、D_m的方程分别为:

$$-\frac{1}{a}\left[-F_n+\pi^2 v\beta_n^2 D\left(B_n-A_n\right)\right]+D(2-v)\pi^2\beta_n^2\left[\frac{B_n-A_n}{a}+\frac{a}{6D}F_n-\frac{\pi^2 v\beta_n^2 a\left(2A_n+B_n\right)}{6}\right]-$$

$$2\beta_n^2\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{m\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)(-1)^{m+1}F_n-$$

$$\frac{2}{n}\sum_{m=1}^{\infty}\frac{\alpha_m^2\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)\left[G_m+(-1)^{n+1}H_m\right]+$$

$$2\pi^2\beta_n^4 D\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{m\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]+$$

$$\frac{2\pi^2 D}{n}\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\alpha_m^4\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+\frac{2}{\beta_n^2 n}\sum_{m=1}^{\infty}\alpha_m^3\left[G_m+(-1)^{n+1}H_m\right]-\frac{2v\pi^2 D}{\beta_n^2 n}\sum_{m=1}^{\infty}\alpha_m^5\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{2D\pi^2}{n}\sum_{m=1}^{\infty}\alpha_m^3\left[C_m+(-1)^{n+1}D_m\right]-\frac{2(-1)^{n+1}(2-v)}{n}\sum_{m=1}^{\infty}\alpha_m\left[G_m-H_m+\pi^2 Dv\alpha_m^2\left(D_m-C_m\right)\right]-$$

$$\frac{2(2-v)\left[1+(-1)^{n+1}\right]}{n}\sum_{m=1}^{\infty}\left(\pi^2 Dv\alpha_m^3 C_m-\alpha_m G_m\right)=0 \tag{23}$$

$$-\frac{1}{a}\left[E_n+\pi^2 v\beta_n^2 D\left(B_n-A_n\right)\right]+D(2-v)\pi^2\beta_n^2\left[\frac{B_n-A_n}{a}-\frac{a}{6D}E_n+\frac{\pi^2 v\beta_n^2 a\left(A_n+2B_n\right)}{6}\right]-$$

$$2\beta_n^2\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{m\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^m\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)E_n-$$

$$\frac{2}{n}\sum_{m=1}^{\infty}\frac{\alpha_m^2\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^m\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)\left[G_m+(-1)^{n+1}H_m\right]+$$

$$2\pi^2\beta_n^4 D\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{m\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^m\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]+$$

$$\frac{2\pi^2 D}{n}\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\alpha_m^4(-1)^m\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\left[\alpha_m^3+(2-v)\beta_n^2\alpha_m\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^m q_{mn}+$$

$$\frac{2}{\beta_n^2 n}\sum_{m=1}^{\infty}\alpha_m^3(-1)^m\left[G_m+(-1)^{n+1}H_m\right]-\frac{2v\pi^2 D}{\beta_n^2 n}\sum_{m=1}^{\infty}\alpha_m^5(-1)^m\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{2D\pi^2}{n}\sum_{m=1}^{\infty}\alpha_m^3(-1)^m\left[C_m+(-1)^{n+1}D_m\right]-$$

$$\frac{2(-1)^{n+1}(2-v)}{n}\sum_{m=1}^{\infty}\alpha_m(-1)^m\left[G_m-H_m+\pi^2 Dv\alpha_m^2\left(D_m-C_m\right)\right]-$$

$$\frac{2(2-v)\left[1+(-1)^{n+1}\right]}{n}\sum_{m=1}^{\infty}(-1)^m\left(\pi^2 Dv\alpha_m^3 C_m-\alpha_m G_m\right)=0 \tag{24}$$

$$-\frac{1}{b}\left[-H_m+\pi^2 v\alpha_m^2 D\left(D_m-C_m\right)\right]+D(2-v)\pi^2\alpha_m^2\left[\frac{D_m-C_m}{b}+\frac{b}{6D}H_m-\frac{\pi^2 v\alpha_m^2 b\left(2C_m+D_m\right)}{6}\right]-$$

$$2\alpha_m^2\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{n\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)(-1)^{n+1}H_m-\frac{2\beta_n^2}{m}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)\left[E_n+(-1)^{m+1}F_n\right]+$$

$$2\pi^2\alpha_m^4 D\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{n\left(\alpha_m^2+\beta_n^2\right)^2}\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{2\pi^2 D}{m}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\beta_n{}^4\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}q_{mn}+$$

$$\frac{2}{\alpha_m^2 m}\sum_{n=1}^{\infty}\beta_n^3\left[E_n+(-1)^{m+1}F_n\right]-\frac{2v\pi^2 D}{\alpha_m^2 m}\sum_{n=1}^{\infty}\beta_n^5\left[A_n+(-1)^{m+1}B_n\right]+\frac{2D\pi^2}{m}\sum_{n=1}^{\infty}\beta_n^3\left[A_n+(-1)^{m+1}B_n\right]-$$

$$\frac{2(-1)^{m+1}(2-v)}{m}\sum_{n=1}^{\infty}\beta_n\left[E_n-F_n+\pi^2 Dv\beta_n^2\left(B_n-A_n\right)\right]-$$

$$\frac{2(2-v)\left[1+(-1)^{m+1}\right]}{m}\sum_{n=1}^{\infty}\left(\pi^2 Dv\beta_n^3 A_n-\beta_n E_n\right)=0 \tag{25}$$

$$-\frac{1}{b}\left[G_m+\pi^2 v\alpha_m^2 D\left(D_m-C_m\right)\right]+D(2-v)\pi^2\alpha_m^2\left[\frac{D_m-C_m}{b}-\frac{b}{6D}G_m+\frac{\pi^2 v\alpha_m^2 b\left(C_m+2D_m\right)}{6}\right]-$$

$$2\alpha_m^2\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{n\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^n\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)G_m-$$

$$\frac{2\beta_n^2}{m}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^n\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)\left[E_n+(-1)^{m+1}F_n\right]+$$

$$2\pi^2\alpha_m^4 D\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{n\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^n\left[v\left(2+\frac{\alpha_m^2}{\beta_n^2}\right)-1\right]\left[C_m+(-1)^{n+1}D_m\right]+$$

$$\frac{2\pi^2 D}{m}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}\beta_n^4(-1)^n\left[v\left(2+\frac{\beta_n^2}{\alpha_m^2}\right)-1\right]\left[A_n+(-1)^{m+1}B_n\right]+$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left[\beta_n^3+(2-v)\alpha_m^2\beta_n\right]}{\left(\alpha_m^2+\beta_n^2\right)^2}(-1)^n q_{mn}+\frac{2}{\alpha_m^2 m}\sum_{n=1}^{\infty}\beta_n^3(-1)^n\left[E_n+(-1)^{m+1}F_n\right]-$$

$$\frac{2v\pi^2 D}{\alpha_m^2 m}\sum_{n=1}^{\infty}\beta_n^5(-1)^n\left[A_n+(-1)^{m+1}B_n\right]+\frac{2D\pi^2}{m}\sum_{n=1}^{\infty}\beta_n^3(-1)^n\left[A_n+(-1)^{m+1}B_n\right]-$$

$$\frac{2(-1)^{m+1}(2-v)}{m}\sum_{n=1}^{\infty}\beta_n(-1)^n\left[E_n-F_n+\pi^2 Dv\beta_n^2\left(B_n-A_n\right)\right]-$$

$$\frac{2(2-v)\left[1+(-1)^{m+1}\right]}{m}\sum_{n=1}^{\infty}(-1)^n\left(\pi^2 Dv\beta_n^3 A_n-\beta_n E_n\right)=0 \tag{26}$$

4 正确性验证计算

4.1 计算步骤

利用前述公式进行板结构计算的步骤为:

1)获取板的计算参数.

2)确定板左下、右下、左上、右上角点的沉降(w_{00}、w_{a0}、w_{0b}、w_{ab}):若某角点无沉降或自由沉降时,取0;有沉降时,取设定的沉降值.

3)设定采用的傅立叶级数的项数,一般不小于40项[7].

4)根据板的边界条件组成求解待定系数的方程组:

板左边:为简支边时,取A_n=E_n=0;为固支边时,选择公式(19)组入方程组;为自由边时,选择公式(23)组入方程组.

板右边:为简支边时,取B_n=F_n=0;为固支边时,选择公式(20)组入方程组;为自由边时,选择公式(24)组入方程组.

板下边:为简支边时,取C_n=G_n=0;为固支边时,选择公式(21)组入方程组;为自由边时,选择公式(25)组入方程组.

板上边:为简支边时,取D_n=H_n=0;为固支边时,选择公式(22)组入方程组;为自由边时,选择公式(26)组入方程组.

5)求解方程组,得到待定系数A_n、B_n、C_m、D_m以及E_n、F_n、G_m、H_m.

6)按公式(10)计算板挠度的傅立叶系数w_{ij}.

7)如果有自由沉降的角点,计算角点处单位面积的弯曲变形能u:

$$u=\frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-v)\left[\frac{\partial^2 w}{\partial x^2}\times\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\} \tag{27}$$

当弯曲变形能u达到极小值,转步骤8);否则设定新的角点沉降值,转步骤4).

8)按公式(7)计算挠度w,按公式(13)~(16)分别计算弯矩M_x、M_y和剪力V_x、V_y.

后面的计算中,计算程序用C语言编写,采用克劳特(Crout)分解法进行线性方程组的求解,弯曲变形能极小值的搜索采用拉格朗日一元三点等距插值法[15].级数解计算中级数项数取80项.有限元数值解采用ADINA软件三角形板单元0.1 m网格进行.

4.2 验证计算结果与对比

选择4种支承条件板进行对比计算.取板尺寸$a\times b$=4 m×4 m,厚h=0.1 m;板弹性模量E=3×10^7 kPa,满布均布荷载q=25 kPa.

4.2.1 一边固支三边自由板

对一边固支三边自由板,在两个自由边相交的角点自由沉降时,可视为悬臂梁.为与梁对比,取泊松比v=0,计算结果列于表1.

表1 与悬臂梁计算结果对比

Tab.1 Comparison of computed results with that of a cantilever

计算方法	板端挠度/mm	固支边中点弯矩/(kN·m)
按悬臂梁计算	320	200
本文方法	317.85	197.40

由表1看出,两种方法的结果一致.挠度的偏差为0.676%,弯矩的偏差为1.317%.

4.2.2　两邻边简支两邻边自由板及四边自由板

文献[16]中有两邻边简支两邻边自由板及四边自由板(在自由边相交的角点均有支承)的计算用表.为便于与文献[16]对比,取泊松比v=0.3,计算结果列于表2.

由表2看出,两种方法的结果也是一致的.最大偏差出现在两邻边简支两邻边自由板的自由边中点挠度,为0.429%.

表2　与文献[16]计算结果对比

Tab.2　Comparison of computed results with that in reference [16]

板支承形式	计算方法	板中心挠度/mm	板中心弯矩/(kN·m)	自由边中点挠度/mm	自由边中点弯矩/(kN·m)
两邻边简支	文献[16]查表	28.817	29.040	32.079	47.120
两邻边自由	本文方法	28.734	29.008	31.942	46.931
四边自由四角点支承	文献[16]查表	59.428	44.680	41.350	60.200
	本文方法	59.263	44.604	41.214	59.972

4.2.3　两邻边固支两邻边自由板

对两邻边固支两邻边自由板在自由边相交的角点自由沉降情况下进行对比计算.取泊松比v=0.3,计算结果列于表3.

由表3看出,两种方法的结果基本是一致的.最大偏差出现在自由边中点弯矩,为3.777%;其次是固支边中点弯矩,为3.435%;其余值的偏差则在1.63%以下.由此可见,本文级数解与解析解、静力计算手册及有限元数值解均符合较好.

表3　与有限元计算结果对比

Tab.3　Comparison of computed results with that by FEM

计算方法	板中心挠度/mm	自由角点挠度/mm	固支边中点弯矩/(kN·m)	自由边中点挠度/mm	自由边中点弯矩/(kN·m)
有限元	20.273	101.68	53.508	46.497	4.140 1
本文方法	20.075	100.05	51.731	45.892	4.302 6

5　集中荷载作用点的弯矩计算

众所周知,在集中荷载情况下,于集中荷载作用处级数解的挠度收敛而弯矩不收敛.对于这一弯矩不收敛的问题,我们提出可利用集中荷载作用点及邻近点的挠度计算值采用差分方法计算来解决.

为保证结果的正确性,差分步长可尽量选得小一些.但鉴于计算机截断误差的限制,差分步长又不能取得过小.我们采用不同的差分步长,与有限元结果对比来确定一个合理的差分步长.选择4种支承条件板,取板尺寸4 m×4 m,厚0.1 m,板弹性模量3×10^7 kPa,泊松比0.3,集中荷载25 kN,作用于板中心.

经采用2~20 mm步长按中心差分公式计算,差分步长的合适取值为10 mm.采用该差分步长的计算结果列于表4.由表4可知,两种方法的偏差较小,说明利用挠度计算集中荷载作用点处的弯矩是简便可行的.

表4　受集中荷载作用板计算结果对比

Tab.4　Comparison of computed results for a plate under a concentrated load

板支承形式	计算方法	板中心弯矩/(kN·m)	相对偏差/%
四边简支	有限元	13.549	2.652
	本文方法	13.199	
四边固支	有限元	12.207	2.961
	本文方法	11.856	
四边自由	有限元	15.602	2.402
	本文方法	15.236	
对边固支一边简支一边自由	有限元	12.863	2.756
	本文方法	12.518	

6　双向板简化为单向板分析

利用本文方法可方便地计算各种支承条件下,随板长宽比的变化,双向板与单向板(即梁)内力的差异.

取矩形板短边方向的长度4 m,长边方向的长度根据长宽比的变化而改变,板厚0.1 m;板弹性模量3×10^7 kPa,为与梁对比取泊松比0;承受满布均布荷载25 kPa.

计算17种支承条件下,板与梁弯矩相差要达到5%及1%时的长宽比,结果列于表5.

根据表5可知:

表5 长宽比计算结果

Tab.5 Computed results of aspect ratio

算例序号	板支承形式	板简化为梁的形式	梁弯矩取值位置	梁弯矩值/(kN·m)	板与梁弯矩要求达到的相对偏差/%	板需要达到的长宽比
1	四边简支	两端简支	中点	50	5	3.16∶1
					1	4.34∶1
2	四边固支	两端固支	端点	$-\frac{100}{3}$	5	1.66∶1
					1	1.95∶1
3	对边固支 对边简支	两端简支	中点	50	5	3.57∶1
					1	4.75∶1
		两端固支	端点	$-\frac{100}{3}$	5	1.29∶1
					1	1.54∶1
4	对边固支对边自由	两端固支	端点	$-\frac{100}{3}$	1	1∶1
5	对边简支对边自由	两端简支	中点	50	1	1∶1
6	对边自由一边简支一边固支	一简一固	固支端	−50	1	1∶1
7	三边简支一边固支	两端简支	中点	50	5	3.37∶1
					1	4.55∶1
		一简一固	固支端	−50	5	1.82∶1
					1	2.35∶1
8	三边固支一边简支	两端固支	端点	$-\frac{100}{3}$	5	1.48∶1
					1	1.74∶1
		一简一固	固支端	−50	5	2.21∶1
					1	2.78∶1
9	三边简支一边自由	两端简支	中点	50	5	1.49∶1
					1	2.07∶1
10	三边固支一边自由	两端固支	端点	$-\frac{100}{3}$	1	1∶1
		悬臂梁	固支端	−200	5	4.95∶1
					1	7.09∶1
11	对边简支一边固支一边自由	两端简支	中点	50	5	1.69∶1
					1	2.29∶1
		悬臂梁	固支端	−200	5	4.18∶1
					1	6.29∶1
12	对边固支一边简支一边自由	两端固支	端点	$-\frac{100}{3}$	1	1∶1
13	两邻边固支两邻边简支	一简一固	固支端	−50	5	2.02∶1
					1	2.57∶1
14	两邻边固支一边简支一边自由	一简一固	固支端	−50	1	1∶1
		悬臂梁	固支端	−200	5	4.57∶1
					1	6.69∶1
15	两邻边简支一边固支一边自由	一简一固	固支端	−50	1	1∶1
16	两邻边固支两邻边自由	悬臂梁	固支端	−200	5	1.65∶1
					1	1.98∶1
17	两邻边自由一边固支一边简支	悬臂梁	固支端	−200	5	1.21∶1
					1	1.58∶1

注:带一个自由边板简化成两端支承梁时,取值位置在弯矩最大的自由边处;其余情况在(绝对值的)最大值处取值.

1)对边支承对边自由板(算例4、5、6):可视作单向板,无长宽比的要求.

2)四边支承板(算例1、2、3、7、8、13):若要简化为两端固支、一端简支一端固支、两端简支单向板,长宽比要分别达到2∶1、2.5∶1、4.5∶1.

3)三边支承一边自由板(算例9、10、11、12、14、

15):若要简化为两端固支、一端简支一端固支、两端简支单向板,(按自由边处的内力来说)长宽比要分别达到1∶1、1∶1、2∶1;若要简化为悬臂单向板,长宽比要达到6∶1.

4)两邻边支承两邻边自由板(算例16、17):若要简化为悬臂单向板,支承边为固支时,(按自由边处的内力来说)长宽比要达到2∶1;支承边为一边简支一边固支时,长宽比要达到1.5∶1.

顺便提及,考虑表1的计算结果,一边固支三边自由板且无长宽比的要求,可视为单向板.

7 结 论

采用带补充项的挠度函数,解决四边不同支承条件下矩形板的弯曲变形问题.给出不同荷载作用下简支边、固支边和自由边任意组合情况下的统一计算公式.

计算表明,利用挠度值按差分法计算弯矩,可避免集中荷载作用处弯矩级数解不收敛的问题.

对17种支承条件板计算出双向板简化为单向板所需要达到的长宽比,计算结果可为工程技术人员参考使用.

值得注意的是,本文只给出4种荷载的傅立叶系数[公式(3)~(6)].要处理其他荷载形式,只需更换成相应的傅立叶系数即可.

参考文献

[1] TIMOSHENKO S, WOINOWSKY K S. Theory of Plates and Shells[M]. 2nd ed. New York: McGraw—Hill Book Company, Inc.,1959:46-47, 82, 180-228.

[2] 严宗达. 结构力学中的富里叶级数解法[M]. 天津: 天津大学出版社, 1989: 153-158.
YAN Z D. Fourier series solution in structural mechanics[M]. Tianjin:Tianjin University Press,1989:153-158.(in Chinese)

[3] 钟万勰, 姚伟岸. 板弯曲求解新体系及其应用[J]. 力学学报, 1999, 31(2): 173-184.
ZHONG W X, YAO W A. A new system for solving plate bending and its application [J]. Chinese Journal of Theoretical and Applied Mechanics,1999, 31(2): 173-184.(in Chinese)

[4] LI R, ZHONG Y, TIAN B, et al. On the finite integral transform method for exact bending solutions of fully clamped orthotropic rectangular thin plates[J]. Applied Mathematics Letters, 2009, 22(12): 1821-1827.

[5] 许琪楼, 姜锐, 唐国明,等. 四边支承矩形板弯曲统一求解方法—兼论纳维叶解与李维解法的统一性[J]. 工程力学, 1999, 16(3): 90-99.
XU Q L, JIANG R, TANG G M, et al. Unified solution method on rectangular plate bending with four edges supported(discussion on unification of the navier solution and levy solutions) [J]. Engineering Mechanics,1999,16(3):90-99.(in Chinese)

[6] 杨端生, 黄炎. 矩形板结构的弯曲问题[J]. 湖南大学学报(自然科学版), 2004, 31(6): 65-69.
YANG D S, HUANG Y. The problem of bending of rectangular plate structure [J]. Journal of Hunan University (Natural Sciences),2004, 31(6): 65-69.(in Chinese)

[7] 杨成永, 马文辉, 韩薛果, 等. 局部均布荷载作用下四边支承矩形板的内力计算[J]. 湖南大学学报(自然科学版), 2020, 47(11): 114-119.
YANG C Y, MA W H, HAN X G, et al. Internal force calculation of four edges supported rectangular plates under local uniformly distributed load [J]. Journal of Hunan University (Natural Sciences), 2020, 47(11): 114-119.(in Chinese)

[8] 钟阳, 李锐, 刘月梅. 四边固支矩形弹性薄板的精确解析解[J]. 力学季刊, 2009, 30(2): 297-303.
ZHONG Y, LI R, LIU Y M. Exact analytic solution of rectangular thin plate with four edges clamped [J]. Chinese Quarterly of Mechanics,2009,30(2):297-303.(in Chinese)

[9] 张景辉. 弹性矩形板动静力问题解析求解[D]. 大连:大连理工大学, 2020.
ZHANG J H. Analytical solution of dynamic and static problems of elastic rectangular plates [D]. Dalian: Dalian University of Technology, 2020.(in Chinese)

[10] 陈英杰, 阙春发, 郭敦. 混合能量原理求解矩形板在静水压力作用下的弯曲[J]. 力学与实践, 2021, 43(4):567-575.
CHEN Y J, QUE C F, GUO D. Determination of the bending of rectangular plate under hydrostatic pressure based on hybrid energy principle[J], Mechanics in Engineering, 2021, 43(4): 567-575.(in Chinese)

[11] 李状飞, 寇子琦, 刘海, 等. 矩形板受分布荷载作用下的解析解[J]. 青岛理工大学学报, 2021, 42(3): 28-35.
LI Z F, KOU Z Q, LIU H, et al. Analytic solution of rectangular plate under distributed loads[J]. Journal of Qingdao University of Technology, 2021, 42(3): 28-35.(in Chinese)

[12] 李传才, 向贤华, 张欣. 混凝土结构单向板与双向板区分界限的研究[J]. 土木工程学报, 2006, 39(3): 62-67.
LI C C, XIANG X H, ZHANG X. The criterion of distinguishing one-way and two-way concrete slabs [J] . China Civil Engineering Journal, 2006,39(3): 62-67.(in Chinese)

[13] 冯霞, 占玉林, 邹赵勇, 等. 钢-混凝土组合桥面单向板与双向板边长比界限研究[J]. 建筑结构学报, 2015, 36(增刊1): 156-160.
FENG X, ZHAN Y L, ZOU Z Y, et al. Analysis on length ratio demarcation between one-way and two-way deck of steel-concrete composite bridge deck [J]. Journal of Building Structures,2015, 36(Sup.1): 156-160.(in Chinese)

[14] CHEN W F, LIU E M. Handbook of Structural Engineering[M]. 2nd ed. New York: CRC Press, 2005: 241-243.

[15] 刘德贵. FORTRAN算法汇编-第一分册[M]. 北京:国防工业出版社,1980: 21-24,183-187.
LIU D G. FORTRAN algorithm assembly-the first volume[M]. Beijing:National Defense Industry Press,1980: 21-24,183-187.(in Chinese)

[16] 《建筑结构静力计算手册》编写组. 建筑结构静力计算手册[M]. 北京: 中国建筑工业出版社, 1998: 223-225.
Working Group of Handbook of Building Structural Statics. Handbook of building structural statics [M]. Beijing: China Architecture & Building Press,1998:223-225.(in Chinese)