DOI: 10.19884/j.1672-5220.202506005

Long-Term Returns in CIR Models Driven by α-Stable Lévy Process

LU Wenhan, YAN Litan*, TAO Qing
School of Mathematics and Statistics, Donghua University, Shanghai 201620, China

Abstract: Interest rates play a crucial role in national economic operations, capital flows, asset pricing, and insurance product returns, yet their long-term dynamics under non-Gaussian noise remain underexplored. This study extends the traditional Cox-Ingersoll-Ross (CIR) model by introducing an α-stable Lévy process and investigates whether the long-term time average of the short-term interest rate converges to a finite limit in the long run under the stable CIR (SCIR) model. Numerical simulations are conducted across various parameter sets to validate the robustness of the model's convergence behavior under different scenarios, and an empirical analysis using data from the Shanghai interbank offered rate (Shibor) is performed. The results confirm that under suitable conditions, the long-run time average of the short-term interest rate under the SCIR model converges to a finite limit, supported by both simulations and Shibor data. These findings provide theoretical insights into the long-term dynamics of interest rates driven by non-Gaussian noise, with practical implications for financial asset pricing and risk management.

Keywords: long-term return; Cox-Ingersoll-Ross (CIR) model; α-stable Lévy process; convergence with probability one
CLC number: O211.9 **Document code**: A
Article ID: 1672-5220(2026)05-0143-11

0 Introduction

Interest rates play a crucial role in the operation of the national economy, directly influencing market capital flows and resource allocation. For the pricing of financial assets, the interest rate is an indispensable factor. Accurate understanding and forecasting of interest rate movements are essential for the proper evaluation of financial asset values. Concurrently, with increasing market competition, companies are paying more attention to customer demands, and customers expect to achieve the highest possible returns. Consequently, insurance companies offer a certain percentage of interest on their products (such as bonds and life insurance).

Let (Ω, F, P) be a probability space, where Ω is the sample space; F is the sigma-algebra of measurable events; P is the probability measure. We consider the stochastic differential equation on the probability space in the following abstract form, which serves as a generalized representation of the stable Cox-Ingersoll-Ross (SCIR) model:

$$X_\alpha(t) = X_\alpha(0) + \int_0^t \{2\beta X_\alpha(s) + \delta(s)\}\,\mathrm{d}s + \int_0^t g(X_\alpha(s-))\,\mathrm{d}M_\alpha(s), \tag{1}$$

where $X_\alpha(t)$ is the state variable at time t; $X_\alpha(0)$ is the initial state of the process; $\beta < 0$ is the drift parameter, in the sequel, t and s are time variables satisfying $s \leqslant t$ (the same understanding applies without further mention); $s-$ denotes the left limit at s; $\delta(\cdot)$, $g(\cdot)$, and $M_\alpha(\cdot)$ are the drift process, diffusion process, and noise process, respectively, and all of them are defined on the same probability space and their precise definitions are given in C1–C3.

C1: $\delta_1 = \{\delta(t),\ t \geqslant 0\}$ is a positive adapted process.

C2: when $\alpha \in (1, 2)$, M_α is a spectrally positive α-stable Lévy process.

$$\nu_\alpha(\mathrm{d}z) = -\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} z^{-\alpha-1}\mathbb{I}_{|z\geqslant 0|}\,\mathrm{d}z,$$

where ν_α is the Lévy measure; z is the integration variable; $\Gamma(\cdot)$ is the Gamma function; $\mathbb{I}$ is an indicator function; α is the stability index.

C3: the function $g(x): \mathbb{R} \to \mathbb{R}_+$ satisfies $g(0) = 0$; $|g(x) - g(y)| \leqslant C\sqrt[\alpha]{|x-y|}$, and $x, y \in \mathbb{R}$; C is a constant, and $C \geqslant 0$.

Then, Eq. (1) has a unique positive (strong) solution[1-2], which is an extension of the pure-jump SCIR model

$$\mathrm{d}X_\alpha(t) = (a - bX_\alpha(t))\,\mathrm{d}t + \sigma\sqrt[\alpha]{X_\alpha(t-)}\,\mathrm{d}M_\alpha(t), \tag{2}$$

where a is the drift constant associated with the long-term mean; b is the speed of mean reversion; σ is the scale

Received date: 2025-06-09
Foundation items: National Natural Science Foundation of China (Nos. 11971101 and 12171081); Shanghai Natural Science Foundation, China (No. 24ZR1402900)
* Correspondence should be addressed to YAN Litan, email: litanyan@dhu.edu.cn
Citation: Lu W H, Yan L T, Tao Q. Long-term returns in CIR models driven by α-stable Lévy process [J]. *Journal of Donghua University (English Edition)*, 2026, 43(5): 143-153.

index; $\alpha \in (1, 2)$, $a > 0$, $b > 0$ and $\sigma > 0$. Further studies on Eq. (2) can be found in Refs. [3–7].

When $\alpha = 2$, Eq. (1) was first considered by Deelstra et al.[8] as a stochastic interest rate model, which is an extension of the Cox-Ingersoll-Ross (CIR) model:

$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}\,dB(t)$ with $a > 0$, where $B(t)$ is a standard Brownian motion; $r(t)$ is a stochastic interest process.

As is known to all, for the instantaneous interest rate process $\{X(t), t \geqslant 0\}$, the mean

$$\frac{1}{t}\int_0^t X(s)\,ds$$

stands for the long-term return; t and s are time variables satisfying $s \leqslant t$. Insurance companies promise a certain fixed percentage of interest on their insurance products such as bonds, life insurance, etc. We wonder how this percentage should be determined. It must be lower than the estimated random return to cover expenses and to hold some reserves so that the company can get through difficult periods of economic crisis without bankruptcy. In this light, we think it is interesting to study and model the long-term return in a mathematical way. Cox et al.[9] introduced the short interest rate model above. It is also a known fact that this model has some empirically relevant properties. In this model, $r(t)$ never becomes negative, and when $2ab > \sigma^2$, $r(t)$ does not reach zero. When $a > 0$ and $b \geqslant 0$, the randomly moving interest rate is mean-reverting toward the long-term constant value b. Based on this, Deelstra et al.[8] considered Eq. (1) with $\alpha = 2$.

Moreover, it is well-known that many financial processes exhibit characteristics such as discontinuity and heavy-tailed distributions[10-11]. These features may limit the ability of the classical CIR model to capture jumps and heavy-tailed behavior. Therefore, in the field of mathematical finance, Lévy processes are frequently used as a generalization of Brownian motion to more accurately describe these financial process characteristics. A natural generalization of Eq. (1) with $\alpha = 2$ is the stochastic differential equation

$$dX(t) = \{2\beta X_\alpha(t) + \delta(t)\}\,dt + g(X(t-))\,dL(t),$$

where $L(t)$ is a Lévy process. In the present paper, we consider Eq. (1). Our first objective is to present and prove the following theorem, which characterizes the long-term return of stochastic interest rate models.

Theorem 1 Let $X_\alpha = \{X_\alpha(t), t \geqslant 0\}$ be the solution of Eq. (1) with the conditions C1–C3 and let there exist a positive random variable $\bar{\delta}$ such that

$$\frac{1}{t}\int_0^t \delta(s)\,ds \to \bar{\delta}$$

almost surely (a. s.) as t tends to infinity. Then, we have

$$\frac{1}{t}\int_0^t X_\alpha(s)\,ds \to \frac{\bar{\delta}}{-2\beta}$$

a. s. as t tends to infinity. Moreover, when $\int_0^t E(\delta(s))\,ds < \infty$ for all $t \geqslant 0$, we then have

$$E(X_\alpha(t)) = e^{2\beta t}E(X_\alpha(0)) + e^{2\beta t}\int_0^t e^{-2\beta s}E(\delta(s))\,ds$$

for all $t \geqslant 0$ and

$$E(\sup_{0\leqslant s\leqslant t} X_\alpha(s)) \leqslant C_\alpha(E(X_\alpha(0)) + \int_0^T E(\delta(r))\,dr + t^{1/(\alpha-1)})$$

for all $t \in [0, T]$ and $T > 0$.

To comprehensively evaluate the dynamic properties and practical potential of the proposed SCIR model, this study conducts numerical analysis in two parts: numerical simulation and empirical investigation. In the numerical simulation section, we set the model parameters as $\sigma = 0.01$, $\beta = -0.01$, and $\delta = 0.05$, and then simulate the numerical solution of the SCIR model under the regime of exponential stability with $1 < \alpha \leqslant 2$. The objective is to examine the model's convergence behavior toward a long-term mean. By comparing the theoretically predicted stationary values with the simulation outcomes, we verify the model's stability under different values of the stability index α. In the empirical analysis section, we select the Shanghai interbank offered rate (Shibor) as the subject of study. Based on the SCIR model framework developed in this paper, we perform parameter estimation and model fitting using daily Shibor data. The estimated parameters are then used to simulate the long-term limiting behavior of the SCIR model, which is subsequently compared with the actual evolution of Shibor. This comparison serves to assess the explanatory power and practical applicability of the model in capturing the dynamics of financial interest rate data.

The remainder of this paper is organized as follows. Section 1 introduces the fundamental theories and mathematical tools related to α-stable processes, which lays the theoretical foundation for the subsequent model analysis. Section 2 is devoted to a rigorous proof of the main Theorem 1, demonstrating its applicability and convergence properties within the framework of the proposed model. Section 3 presents both numerical simulations and empirical studies based on real financial data, aiming to evaluate the practical performance and effectiveness of the SCIR model. Section 4 concludes the study by summarizing the theoretical and practical contributions of the SCIR model to long-term interest rate modeling, offering insights for financial forecasting, risk management, and policy analysis.

1 Preliminaries

In this section, we briefly recall some basic results of α-stable motion and distribution. We refer to Refs. [12–14] and the references therein. Throughout this paper, we work with a complete probability space, and assume that C stands for a positive constant depending only on the subscript, and its value may be different in

different appearances.

The parameters α, σ, β_1 and μ satisfy

$$\alpha \in (0, 2], \sigma \in (0, +\infty),$$
$$\beta_1 \in [-1, 1], \mu \in (-\infty, +\infty),$$

where β_1 and μ are the skewness index and location index, respectively.

$$\psi_\alpha(u) = \begin{cases} \sigma^\alpha |u|^\alpha \left(1 - i\beta_1 \operatorname{sgn}(u) \tan\frac{\pi\alpha}{2}\right) - i u\mu, & \text{if } \alpha \neq 1, \\ \sigma |u| \left(1 + i\beta_1 \frac{2}{\pi} \operatorname{sgn}(u) \ln|u|\right) - i u\mu, & \text{if } \alpha = 1. \end{cases}$$

where $u \in (-\infty, +\infty)$ is the characteristic argument; $\operatorname{sgn}(u)$ denotes the sign function and $i^2 = -1$. A random variable η admits an α-stable distribution if it has the characteristic function

$$E(e^{iu\eta}) = e^{-\psi_\alpha(u)},$$

where $\eta \sim S_\alpha(\sigma, \beta_1, \mu)$. The function ψ_α is called the Lévy symbol.

A process $Z = \{Z_t, t \geqslant 0\}$ is called an α-stable Lévy process or an α-stable motion, t_1, and t_2 are time variables satisfying $t_2 \leqslant t_1$, and Z is adapted to the filtration $\mathcal{G} = \{G_t, t \geqslant 0\}$ on the above probability space

$$E(e^{iu(Z_{t_1} - Z_{t_2})} \mid \mathcal{G}_{t_2}) = e^{-(t_1 - t_2)\psi_\alpha(u)}$$

for all $t_1 > t_2 \geqslant 0$. The α-stable stochastic integral

$$Y_u = \int_0^u \phi_t \, dZ_t, \ u \geqslant 0,$$

exists for $\alpha \in (0, 2)$ and any adapted process $\phi = \{\phi_t, t \geqslant 0\}$ satisfying $\int_0^T |\phi_t|^\alpha dt < \infty$, a. s., $\forall T > 0$. Moreover, for any ϕ, t_1, T are time variables satisfying $t_1 \leqslant T$, we have that

$$E\left(\sup_{0 \leqslant t_1 \leqslant T} \left| \int_0^{t_1} \phi_t dZ_t \right|^p\right) \leqslant C_{\alpha, p} \left(\int_0^T E(|\phi_t|^\alpha) dt\right)^{p/\alpha},$$

where $C_{\alpha, p}$ is a constant that depends on the values of α and $1 < p < \alpha$ for all $T > 0$, provided that the right side is finite.

In this paper, we only consider a spectrally positive α-stable motion $M_\alpha = \{M_\alpha(t), t \geqslant 0\}$ with $\alpha \in (1, 2)$, $\beta_1 = 1$, $\sigma = 1$ and $\mu = 0$. Thus, M_α is strictly stable and admits the representation

$$M_\alpha(t) = \int_0^s \int_0^\infty x \tilde{N}(dt, dx),$$

where $\tilde{N}$ is a compensated Poisson random measure associated with N defined as

$$\tilde{N}(dt, dx) = N(dt, dx) + \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)\cos(\pi\alpha/2)} x^{-\alpha - 1} 1_{\{x \geqslant 0\}} dx dt.$$

For the spectrally positive α-stable motion, $M_\alpha = \{M_\alpha(t), t \geqslant 0\}$, we have the following strong law of large numbers[15].

2 Theorem 1: Statement and Proof

2.1 Lemma

Let $X_\alpha = \{X_\alpha(t), t \geqslant 0\}$ be the unique positive solution to Eq. (1). For simplicity, we also assume that C denotes a positive constant depending on the subscripts and its value may be different in different appearances. The case $\alpha = 2$ has been proven in Ref. [8], where we only need to consider $1 < \alpha < 2$.

Lemma 1 Bihari-LaSalle's inequality

Let $t \longmapsto w(t)$ be a continuous (increasing, right-continuous) function on $[0, \infty)$ and let $t \longmapsto v(t)$ be a continuous function on $[0, \infty)$. If $x \longmapsto h(x)$ is an increasing function on $[0, \infty)$ and

$$w(t) \leqslant a + b\int_0^t v(s) h(w(s)) ds,$$

where $w(\cdot)$ is the function to be estimated; $v(\cdot)$ is a given continuous function on $[0, \infty)$; t and s are time variables satisfying $s \leqslant t$ for any $t \in [0, \infty)$; a, $b \geqslant 0$ are both constants, then,

$$w(u) \leqslant H^{-1}(H(a) + b\int_0^u v(s) ds),$$

where x is an integration variable satisfying $x \leqslant a$ for all $a \geqslant 0$; $H(a) = \int_0^a \frac{1}{h(x)} dx$; H^{-1} denotes the inverse function of H.

Lemma 2 Let the conditions in Theorem 1 hold. Then, $\sup_{0 \leqslant s \leqslant T} X_s$ is finite for all $T \geqslant 0$, and when $\int_0^t E\delta(s) ds < \infty$ for all $t \geqslant 0$, we have

$$E(X_\alpha(t)) = e^{2\beta t} E(X_\alpha(0)) + e^{2\beta t} \int_0^t e^{-2\beta s} E(\delta(s)) ds, \quad \forall t \geqslant 0, \tag{3}$$

and

$$E(\sup_{0 \leqslant s \leqslant t} X_\alpha(s)) \leqslant C_\alpha (E(X_\alpha(0)) + \int_0^T E(\delta(r)) dr + t^{1/(\alpha - 1)}), \tag{4}$$

where C_α is a constant that depends on the value of α for all $t \in [0, T]$ and $T > 0$.

The proof for Lemma 2 will be provided in two cases.

Proof of Case I

The integral $\int_0^t E(\delta(s)) ds$ converges for all $t \geqslant 0$.

By using integration by parts in the process $t \longmapsto X_\alpha(t) e^{-2\beta t}$ with $t \geqslant 0$, we have that

$$X_\alpha(t) = e^{2\beta t} X_\alpha(0) + \int_0^t \delta(u) e^{2\beta(t-u)} du + \int_0^t e^{2\beta(t-u)} g(X_\alpha(u)) dM_\alpha(u) \tag{5}$$

for any $0 \leqslant u \leqslant t$. Define the stopping times $\{\tau_n, n \in \mathbb{N}\}$ by

$$\tau_n = \inf\{t \geqslant 0 \| X_\alpha(t)| \geqslant n\}.$$

Then, we have that

$$X_\alpha(t \wedge \tau_n) = \mathrm{e}^{2\beta t\wedge\tau_n}X_\alpha(0) + \int_0^{t\wedge\tau_n}\delta(u)\mathrm{e}^{2\beta(t\wedge\tau_n-u)}\mathrm{d}u + \int_0^{t\wedge\tau_n}\mathrm{e}^{2\beta(t\wedge\tau_n-u)}g(X_\alpha(u))\mathrm{d}M_\alpha(u) \tag{6}$$

for all $n \in \mathbb{N}$. Then, the process $\{X_\alpha(t \wedge \tau_n),\ t \geqslant 0\}$ is bounded, and

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^{t\wedge\tau_n}\mathrm{e}^{-2\beta s}g(X_\alpha(s))\mathrm{d}M_\alpha(s)\right|^p\right) \leqslant C_p\left(\int_0^{T\wedge\tau_n}E(|g(X_\alpha(s))|^\alpha)\mathrm{e}^{-2\alpha\beta s}\mathrm{d}s\right)^{p/\alpha} \leqslant$$
$$C_p\mathrm{e}^{-2p\beta(T\wedge\tau_n)}\left(\int_0^{T\wedge\tau_n}E(X_\alpha(s))\mathrm{d}s\right)^{p/\alpha} \leqslant C_p\mathrm{e}^{-2p\beta(T\wedge\tau_n)}nE(T\wedge\tau_n) < \infty$$

for every $n \in \mathbb{N}$, $1 \leqslant p < \alpha$ and $T > 0$. It follows from Eq. (6), which implies that

$$E\left(\sup_{0\leqslant t\leqslant T}X_\alpha(t \wedge \tau_n)\right) \leqslant E(X_\alpha(0)) + E\left(\sup_{0\leqslant t\leqslant T}\int_0^{t\wedge\tau_n}\delta(u)\mathrm{e}^{2\beta(t\wedge\tau_n-u)}\mathrm{d}u\right) + C_p\mathrm{e}^{-2p\beta(T\wedge\tau_n)}nE(T\wedge\tau_n) \leqslant C_{p,\beta,T} < \infty, \tag{7}$$

for all $n \in \mathbb{N}$ and $T > 0$, which implies that the family $X^{\tau_n} := \{X_\alpha(t \wedge \tau_n), t \in [0,T]\}$, $(n \in \mathbb{N})$ is uniformly integrable. Thus, by taking expectations and limits on the two sides of Eq. (6), we get

$$E(X_\alpha(t)) = \lim_{n\to\infty}E(X_\alpha(t \wedge \tau_n)) = \mathrm{e}^{2\beta t}E(X_\alpha(0)) + \mathrm{e}^{2\beta t}\int_0^t\mathrm{e}^{-2\beta s}E(\delta(s))\mathrm{d}s,$$

for all $t > 0$ and the identity Eq. (3) follows.

We now verify Eq. (4). By Eq. (5), it follows that

$$\begin{aligned}
E\left(\sup_{0\leqslant s\leqslant T}X_\alpha(s)\right) &\leqslant E(X_\alpha(0)) + E\left(\sup_{0\leqslant t\leqslant T}\int_0^s\delta(r)\mathrm{e}^{2\beta(s-r)}\mathrm{d}r\right) + E\left(\sup_{0\leqslant s\leqslant T}\left|\int_0^s\mathrm{e}^{2\beta(s-r)}g(X_\alpha(r))\mathrm{d}M_\alpha(r)\right|\right)\\
&\leqslant E(X_\alpha(0)) + \int_0^T E(\delta(r))\mathrm{d}r + C_\alpha\left(\int_0^T E(|g(X_\alpha(r))|^\alpha)\mathrm{e}^{2\beta\alpha(T-r)}\mathrm{d}r\right)^{1/\alpha}\\
&\leqslant E(X_\alpha(0)) + \int_0^T E(\delta(r))\mathrm{d}r + C_\alpha\left(\int_0^T E(X_\alpha(r))\mathrm{e}^{2\beta\alpha(T-r)}\mathrm{d}r\right)^{1/\alpha}\\
&\leqslant E(X_\alpha(0)) + \int_0^T E(\delta(r))\mathrm{d}r + C_\alpha\left(\int_0^T E(X_\alpha(r))\mathrm{d}r\right)^{1/\alpha} < \infty,
\end{aligned}$$

where C_α is a constant that depends on the value of α for any $T > 0$. It follows that

$$u(t) \leqslant a + b\int_0^t(u(r))^{1/\alpha}\mathrm{d}r$$

for any $T > 0$, where

$$u(t) = E\left(\sup_{0\leqslant s\leqslant t}X_\alpha(s)\right)^\alpha,\ a = 2^{\alpha-1}(E(X_\alpha(0)) + \int_0^T E(\delta(r))\mathrm{d}r)^\alpha,\ b = 2^{\alpha-1}C_\alpha.$$

Thus, by Bihari-LaSalle's inequality, we get

$$u(t) \leqslant H^{-1}(H(a) + b\int_0^u v(s)\mathrm{d}s)$$

for any $t \in [0,\ T]$ and $T > 0$, and Eq. (4) follows. As an immediate result, we have $\sup_{0\leqslant s\leqslant t}X_\alpha(s) < \infty$ for any $t \in [0,\ T]$ and $T > 0$.

Proof of Case Ⅱ

Let us now look at the general case with the local assumption $\int_0^t\delta(s)\mathrm{d}s < \infty$ a. s. Define stopping times σ_n, $n \in \mathbb{N}$ by

$$\sigma_n = \inf\left\{t \geqslant 0 \mid \int_0^t\frac{\delta(r)}{(1+r)^\alpha}\mathrm{d}r \geqslant n\right\},$$

and denote $\delta_n(t) := \delta(t)1_{[0,\sigma_n]}$ for $n \in \mathbb{N}$. The stochastic differential equation

$$\mathrm{d}X_\alpha^n(t) = \{2\beta X_\alpha^n(t) + \delta_n(t)\}\mathrm{d}t + g(X_\alpha^n(s-))\mathrm{d}M_\alpha(s)$$

has a unique solution and

$$\int_0^t E(\delta_n(s))\mathrm{d}s \leqslant (1+t)^\alpha n$$

for $n \in \mathbb{N}$. It follows from Case I that $\sup_{0\leqslant s\leqslant t}X_\alpha^n(s) < \infty$ a. s. for any $t \geqslant 0$. On $[0,\ \sigma_n]$, all X_α^k are equal for $k \geqslant n$ by the

uniqueness of the solution of the stochastic differential equation. Noting that $\cup_n [0, \sigma_n] \supset [0, t]$, we obtain the desired result since $\int_0^t \delta(s)\,\mathrm{d}s < \infty$ for all $t \geqslant 0$.

2.2 Proof for Theorem 1

Now, we verify the convergence in Theorem 1. Dividing Eq. (1) by $t+1$ over the time interval $[0, t]$ yields:

$$\frac{2\beta}{1+t}\int_0^t X_\alpha(s)\,\mathrm{d}s = -\frac{1}{1+t}\int_0^t \delta(s)\,\mathrm{d}s - \frac{1}{1+t}\int_0^t g(X_\alpha(s))\,\mathrm{d}M_\alpha(s) + \frac{X_\alpha(t) - X_\alpha(0)}{1+t},$$

for all $t \geqslant 0$. Thus, to end the proof, it is sufficient to check the convergence.

$$Y_\alpha(t;1) := \frac{1}{1+t}\int_0^t g(X_\alpha(s))\,\mathrm{d}M_\alpha(s) \to 0 \tag{8}$$

and

$$Y_\alpha(t;2) := \frac{X_\alpha(t)}{1+t} \to 0 \tag{9}$$

hold a. s., as t tends to infinity. We verify the two convergences in the next two cases.

Case Ⅰ: For every $t \in [0, \infty)$, the integral $\int_0^t E(\delta(s))\,\mathrm{d}s$ converges.

Case Ⅱ: For every $t \in [0, \infty)$, the integral $\int_0^t \delta(s)\,\mathrm{d}s$ converges a. s.

Proof of Case Ⅰ

To prove the convergence of Eq (8). It follows that

$$E\left(\sup_{T_n \leqslant t < T_{n+1}} \left|\frac{1}{1+t}\int_0^t g(X_\alpha(s))\,\mathrm{d}M_\alpha(s)\right|^p\right) \leqslant \frac{C_p}{T_{n+1}^p}\left(\int_0^{T_{n+1}} E(|g(X_\alpha(s))|^\alpha)\,\mathrm{d}s\right)^{p/\alpha} \leqslant \frac{C_p}{T_{n+1}^p}\left(\int_0^{T_{n+1}} E(X_\alpha(s))\,\mathrm{d}s\right)^{p/\alpha}$$

for all $n \geqslant 1$ and $1 < p < \alpha$, where $T_n = n^{\frac{\alpha}{\alpha-1}}$. Moreover, by Lemma 2 we have

$$\int_0^{T_{n+1}} E(X_\alpha(s))\,\mathrm{d}s = \int_0^{T_{n+1}} \mathrm{e}^{2\beta s} E(X_0)\,\mathrm{d}s + \int_0^{T_{n+1}} \mathrm{e}^{2\beta t}\int_0^s \mathrm{e}^{-2\beta r} E(\delta(r))\,\mathrm{d}r\,\mathrm{d}s = -\frac{1}{2\beta}E(X_0)(1 - \mathrm{e}^{2\beta T_{n+1}}) - \frac{1}{2\beta}\int_0^{T_{n+1}} E(\delta(r))(1 - \mathrm{e}^{2\beta(T_{n+1}-r)})\,\mathrm{d}r \leqslant -\frac{1}{2\beta}E(X_0) - \frac{1}{2\beta}\int_0^{T_{n+1}} E(\delta(r))\,\mathrm{d}r$$

for all $n \in \mathbb{N}$ since $\beta < 0$. Consequently,

$$E\left(\sup_{T_n \leqslant t < T_{n+1}} \left|\frac{1}{1+t}\int_0^t g(X_\alpha(s))\,\mathrm{d}M_\alpha(s)\right|^p\right) \leqslant \frac{C_{\alpha,p}}{(1+T_n)^p}\left(\int_0^{T_{n+1}} E|g(X_\alpha(s))|^\alpha \mathrm{d}s\right)^{p/\alpha} \leqslant \frac{C_{\alpha,p}}{(1+T_n)^{p-\frac{p}{\alpha}}}\left(\frac{1}{T_n}\int_0^{T_{n+1}} E(X_\alpha(s))\,\mathrm{d}s\right)^{p/\alpha} = \frac{C_{\alpha,p}}{(1+n)^p}\left(\frac{1}{T_n}\int_0^{T_{n+1}} E(X_\alpha(s))\,\mathrm{d}s\right)^{p/\alpha} \leqslant \frac{C_{\beta,\alpha,p}}{(1+n)^p}\left(\frac{-E(X_0)}{2\beta T_n} - \frac{1}{2\beta T_n}\int_0^{T_{n+1}} E(\delta(r))\,\mathrm{d}r\right)^{p/\alpha},$$

where $C_{\alpha,p}$ is a constant that depends on the values of p and α; $C_{\beta,\alpha,p}$ is a constant that depends on the values of p, β, and α as $n \in \mathbb{N}$ $(1 < p < \alpha)$. Combining this with

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E(\delta(s))\,\mathrm{d}s = E(\bar{\delta}),$$

we show that the series

$$\sum_{n=0}^{\infty} P\left(\sup_{T_n \leqslant t < T_{n+1}} \left|\frac{1}{1+t}\int_0^t g(X_\alpha(s))\,\mathrm{d}M_\alpha(s)\right| > \varepsilon\right)$$

converges for any $\varepsilon > 0$, and the convergence Eq. (8) follows.

The convergence in Eq. (9) can be proved by applying Eq. (5) with $r = 0$, and we obtain

$$\frac{X_\alpha(t)}{1+t} = \mathrm{e}^{2\beta t}\frac{X_\alpha(0)}{1+t} + \frac{1}{1+t}\mathrm{e}^{2\beta t}\int_0^t \delta(u)\,\mathrm{e}^{-2\beta u}\mathrm{d}u + \frac{1}{1+t}\mathrm{e}^{2\beta t}\int_0^t \mathrm{e}^{-2\beta u} g(X_\alpha(u))\,\mathrm{d}M_\alpha(u)$$

for any $t \geqslant 0$. By Eq. (8), we have

$$\frac{1}{1+t}\mathrm{e}^{2\beta t}\int_0^t \mathrm{e}^{-2\beta u} g(X_\alpha(u))\,\mathrm{d}M_\alpha(u) \to 0$$

a. s., as t tends to infinity. Moreover, for any $\gamma \in (0,1)$ we have

$$\frac{1}{1+t}\mathrm{e}^{2\beta t}\int_0^{t-t^\gamma} \delta(u)\,\mathrm{e}^{-2\beta u}\,\mathrm{d}u \leqslant \frac{1}{1+t}\mathrm{e}^{2\beta t^\gamma}\int_0^{t-t^\gamma} \delta(u)\,\mathrm{d}u \leqslant \frac{1}{1+t}\mathrm{e}^{2\beta t^\gamma}\int_0^{t-t^\gamma} \delta(u)\,\mathrm{d}u \to 0$$

and

$$\frac{1}{1+t}\mathrm{e}^{2\beta t}\int_{t-t^\gamma}^t \delta(u)\,\mathrm{e}^{-2\beta u}\,\mathrm{d}u \leqslant \frac{1}{1+t}\int_{t-t^\gamma}^t \delta(u)\,\mathrm{d}u = \frac{1}{1+t}\int_0^t \delta(u)\,\mathrm{d}u - \frac{1}{1+t}\int_0^{t-t^\gamma} \delta(u)\,\mathrm{d}u \to 0$$

a. s., as t tends to infinity, since $\beta < 0$, $\delta(t) \geqslant 0$ and $\frac{1}{t}\int_0^t \delta(u)\,\mathrm{d}u \to \bar{\delta}$. It follows that

$$\frac{1}{1+t}\mathrm{e}^{2\beta t}\int_0^t \delta(u)\,\mathrm{e}^{-2\beta u}\,\mathrm{d}u = \frac{1}{1+t}\mathrm{e}^{2\beta t}\left(\int_0^{t-t^\gamma} \delta(u)\,\mathrm{e}^{-2\beta u}\,\mathrm{d}u + \int_{t-t^\gamma}^t \delta(u)\,\mathrm{e}^{-2\beta u}\,\mathrm{d}u\right) \to 0$$

a. s., as t tends to infinity. Combining this with Eq. (8), we obtain the convergence of Eq. (9). Thus, we complete the proof of Theorem 1 if $\int_0^t E(\delta(s))\,\mathrm{d}s < \infty$ for all $t \geqslant 0$.

Proof of Case Ⅱ

According to the proof of the convergence of Eq. (9) in Case I, we find that its proof only requires the condition $\int_0^t \delta(s)\,\mathrm{d}s < \infty$ a. s. for all $t \geqslant 0$, so we only need to verify the convergence of Eq. (8). In Case Ⅱ, in the proof of Lemma 2 by considering the stopping times $\{\sigma_n,\ n \in \mathbb{N}\}$, we have that $\int_0^t \delta_n(s)\,\mathrm{d}s < \infty$ a. s. for all $t \geqslant 0$ and

$$\frac{1}{1+t}\int_0^t g(X_\alpha^n(s))\,\mathrm{d}M_\alpha(s) \to 0$$

a. s., as t tends to infinity. It follows that the convergence of Eq. (8) holds on $\{\sigma_n = \infty\}$ for all $n \in \mathbb{N}$. Combining this with the fact that $\{\sigma_n = \infty\} \uparrow \Omega$ a. s. as t tends to infinity, we obtain the desired convergence of Eq. (8) and the theorem follows if $\int_0^t \delta(s)\,\mathrm{d}s < \infty$ for all $t \geqslant 0$.

3 Numerical Simulations and Empirical Analysis

In the SCIR model constructed in this study, particular attention is paid to the parameter term $\delta(t)$, which determines the long-term mean behavior of the reference interest rate process. While various modeling approaches for $\delta(t)$ are conceivable, such as treating it as an ergodic process or as a continuous process converging to a positive random variable, we adopt the constant specification $\delta(t) \equiv a$ (where a is an unknown positive parameter) as the primary setting, for the following reasons.

First, from an economic perspective, the assumption $\delta(t) \equiv a$ reflects the stability of long-term interest rate expectations. In financial markets, interest rates are typically guided toward a reasonable range through the interaction of supply and demand, which supports the plausibility of this assumption.

Second, from the viewpoint of mathematical tractability and analytical clarity, the constant specification of $\delta(t)$ allows the SCIR model to possess more explicit analytical solutions, with a clearly defined long-term mean. This greatly facilitates theoretical analysis and numerical simulation of the model's convergence behavior. Such a simplification retains the essential capability of capturing interest rate dynamics under α-stable Lévy noise, while reducing the complexity of parameter estimation and computational implementation.

Third, with regard to empirical feasibility and existing literature, Li et al.[6] have systematically studied parameter estimation methods for the case in which $\delta(t)$ is an unknown constant, and have developed estimators suitable for α-stable Lévy-driven models. Building upon this foundation, we incorporate real Shibor data to estimate the model parameters and simulate the SCIR dynamics, comparing the results with actual Shibor trends to assess the model's capability in capturing long-term behaviors.

In summary, the choice to model $\delta(t)$ as a constant is grounded in a deep understanding of the structural characteristics of China's financial markets, while also balancing theoretical tractability and empirical implementability. This assumption allows for a more transparent evaluation of the SCIR model's performance in capturing the long-run dynamics of interbank interest

rates. Moreover, it enhances the model's ability to reflect mean-reversion properties of market interest rates, providing a robust analytical framework for practical financial decision-making.

Specifically, we use MATLAB to simulate the long-term behavior of the SCIR model with different values of α (namely, 4/3, 3/2, and 7/4). Subsequently, we employ Shibor data to empirically analyze the model's long-term convergence characteristics under the same α settings. These analyses aim to quantitatively validate the model's convergence properties under various parameter configurations.

3.1 Numerical simulation

We begin by specifying the model parameters of Eq. (2) as $\sigma = 0.01$, $\beta = -0.01$, and $\delta = 0.05$. To explore the behavior of the SCIR model under different stability indices, we focus on the regime where $1 < \alpha < 2$ and numerically simulate the solution of the model. The simulation is conducted over a fixed time horizon $t = 2\,000$, discretized into $N = 2\,000$ uniform time steps, yielding a fixed time increment of t/N per step. Under these settings, the theoretical long-run equilibrium value of the process can be derived as $(-\delta)/(2\beta) = 2.5$.

As illustrated in Fig. 1, for all three stability indices, namely, $\alpha = 4/3$, $\alpha = 3/2$, and $\alpha = 7/4$, the simulated SCIR trajectories gradually approach and remain close to the theoretical long-run mean of 2.5. Although slight differences in fluctuations can be observed among the three cases, all trajectories exhibit clear mean-reverting behavior and relatively small long-run deviations from the theoretical equilibrium level. These numerical results illustrate the good long-run convergence behavior of the SCIR model for the selected values of α.

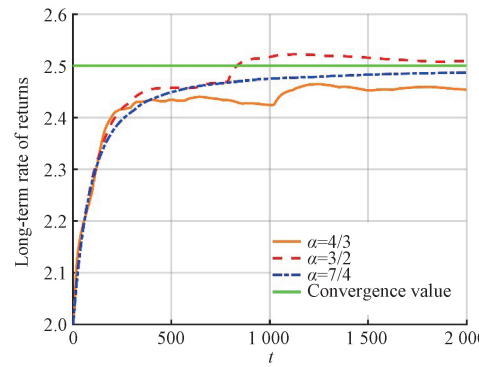


Fig. 1 Comparison of convergence behaviors for different α values in SCIR model

As shown in Fig. 2, the average simulated paths for $\alpha = 4/3$, $\alpha = 3/2$, and $\alpha = 7/4$ exhibit different fluctuation patterns. Despite these differences, all three paths remain close to the theoretical long-run mean, indicating good mean-reverting behavior of the SCIR model.

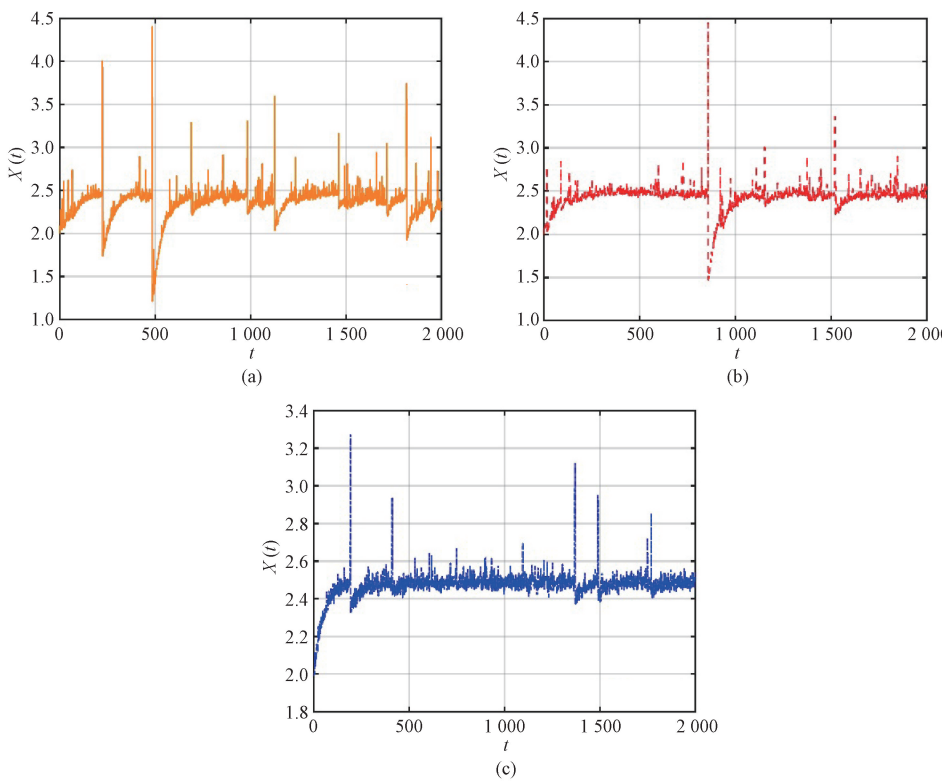


Fig. 2 Solutions of $X(t)$ for different values of α in SCIR model: (a) $\alpha = 4/3$; (b) $\alpha = 3/2$; (c) $\alpha = 7/4$

3.2 Empirical analysis

In this section, we apply the SCIR model in conjunction with parameter estimation and numerical simulations to analyze Shibor. The Shibor data used in

this study are sourced from the RESSET financial database and are utilized to validate the convergence properties of the SCIR model.

During the bank quoting process, each bank submits a lending rate. To compute Shibor, the four highest and four lowest quotes are removed to mitigate the impact of outliers. The remaining quotes are averaged to produce the final Shibor data. This approach ensures accuracy and representativeness, reflecting market liquidity and demand-supply dynamics. Shibor thus offers market participants a reliable reference for investment and financing decisions. By analyzing Shibor, we can assess the long-term convergence behavior of Shanghai's interbank lending rate. Shibor includes multiple maturities, such as overnight, 1-week, 2-week, 1-month, 3-month, 6-month, 9-month, and 1-year, covering various time spans from short to medium-term, providing market participants with diverse rate references.

For this study, we select Shibor as the research subject. After processing the data from October 8, 2006, to August 29, 2025, 4 722 valid overnight Shibor observations are obtained, which will serve as the test set for parameter estimation. The estimated parameters will be applied to the SCIR model to calculate its convergence value. The computed convergence value will then be compared with the actual daily Shibor data trend to validate the effectiveness of the SCIR model. The original Shibor series and its tail characteristics are shown in Fig. 3.

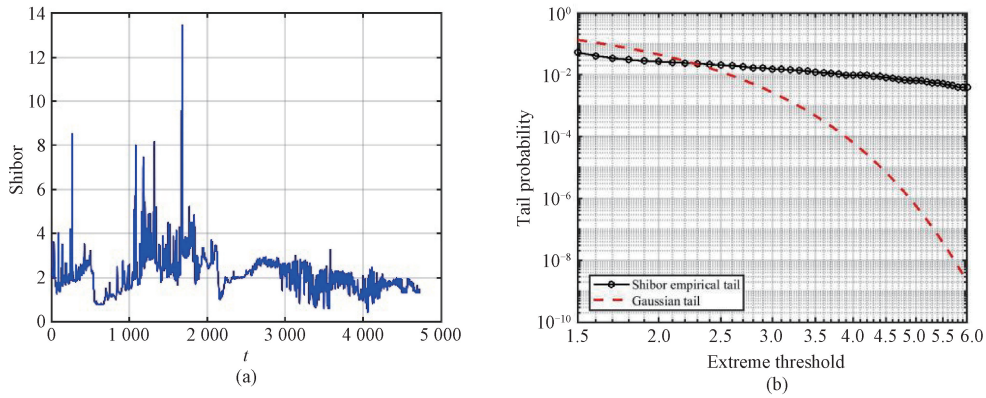


Fig. 3 Empirical characteristics of the overnight Shibor: (a) original Shibor series; (b) tail probability of standardized changes, with Gaussian benchmark for reference

Figure 3 (a) presents the overnight Shibor observations, while Fig. 3(b) compares the empirical tail probabilities of standardized Shibor changes with the Gaussian benchmark. The substantially slower decay of the empirical tail indicates pronounced non-Gaussian, heavy-tailed behavior, suggesting that the Gaussian case may be insufficient to capture the observed tail. Therefore, we focus on the non-Gaussian stable regime.

Before conducting the empirical analysis, it is necessary to estimate the parameters (β, δ_s) in the SCIR model, as they are initially unknown. Li et al.[6] have concluded that both the classical least squares estimators (CLSEs) and the weighted classical least squares estimators (WCLSEs) can be used for parameter estimation, and they have provided analytical expressions for the parameter estimates. Considering the effectiveness of WCLSEs and its stability with a large number of samples $(X_k,\ k = 0,1,\cdots,n)$, this study opts to use WCLSEs for parameter estimation. The estimated parameters $(a,\ b)$ obtained through WCLSEs are as follows.

$$\tilde{b}_n = -\log\frac{\sum_{k=1}^{n} X_k \sum_{k=1}^{n} \frac{1}{X_{k-1}+1} - n\sum_{k=1}^{n}\frac{X_k}{X_{k-1}+1}}{\sum_{k=1}^{n}(X_{k-1}+1)\sum_{k=1}^{n}\frac{1}{X_{k-1}+1} - n^2},$$

and

$$\tilde{a}_n = \frac{\tilde{b}_n\left(\sum_{k=1}^{n} X_k - \mathrm{e}^{-\tilde{b}_n}\sum_{k=1}^{n} X_{k-1}\right)}{n\left(1-\mathrm{e}^{-\tilde{b}_n}\right)}.$$

The parameters a and b are related to the parameters β and δ_s in the SCIR model:

$$\delta_s = a,\ \beta = -b/2.$$

The values of X_k represent the daily Shibor data. After substituting these values, we first compute $(a,\ b)$ and then convert them to β and δ_s. Using MATLAB, the calculated parameter values are

$$a = 0.081\,0,\ b = 0.037\,8,$$

$$\delta_s = a = 0.081\,0,\ \beta = -b/2 = -0.018\,9.$$

The long-term average value of the short-term interest rate, i. e., the long-term return, is computed and its convergence is verified. According to the SCIR model, the long-term return converges to approximately 2.14. This value is in proximity to the average of the Shibor data collected daily from October 8, 2006, to August 29, 2025, which is 2.144 8. The detailed calculation is as follows.

$$\frac{1}{s}\int_0^s X_u\,\mathrm{d}u \to \frac{-\delta}{2\beta} = \frac{-0.081\,0}{2\times(-0.018\,9)} \approx 2.142\,9\ (\text{a.s.}).$$

After obtaining the specific parameter values, we set σ = 0.01 to satisfy the model assumption $\delta_s \geqslant \sigma^2/2$. To ensure consistency in the validation process, we assign values to α as 4/3, 3/2, and 7/4.

Similarly, the simulation time horizon is set to t = 4 722, with N = 4 722 time steps. A fixed time step of t/N is used in the simulation, resulting in a total of 4 722 generated samples.

Figure 4 illustrates the long-run behavior of the SCIR model for α = 4/3, α = 3/2, and α = 7/4. The green line represents the theoretical long-run mean of 2.142 9. For all three values of α, the simulated average paths gradually approach and remain close to this level, although their short-term fluctuations differ slightly. These results indicate that the SCIR model exhibits good mean-reverting behavior and stable long-run performance under the selected stability indices.

The corresponding simulated paths based on the Shibor data are presented in Fig. 5. Although occasional fluctuations and jumps are observed, the trajectories generally remain around the estimated long-run level. This behavior is consistent with the convergence pattern shown in Fig. 4 and further illustrates the ability of the SCIR model to capture both the long-run mean-reverting behavior and short-term fluctuations of the Shibor data.

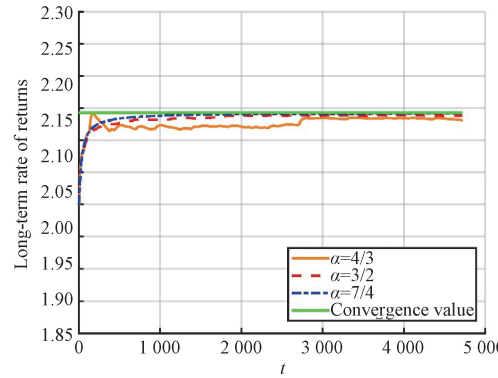


Fig. 4 Comparison of convergence behaviors for different α values in SCIR model using Shibor data

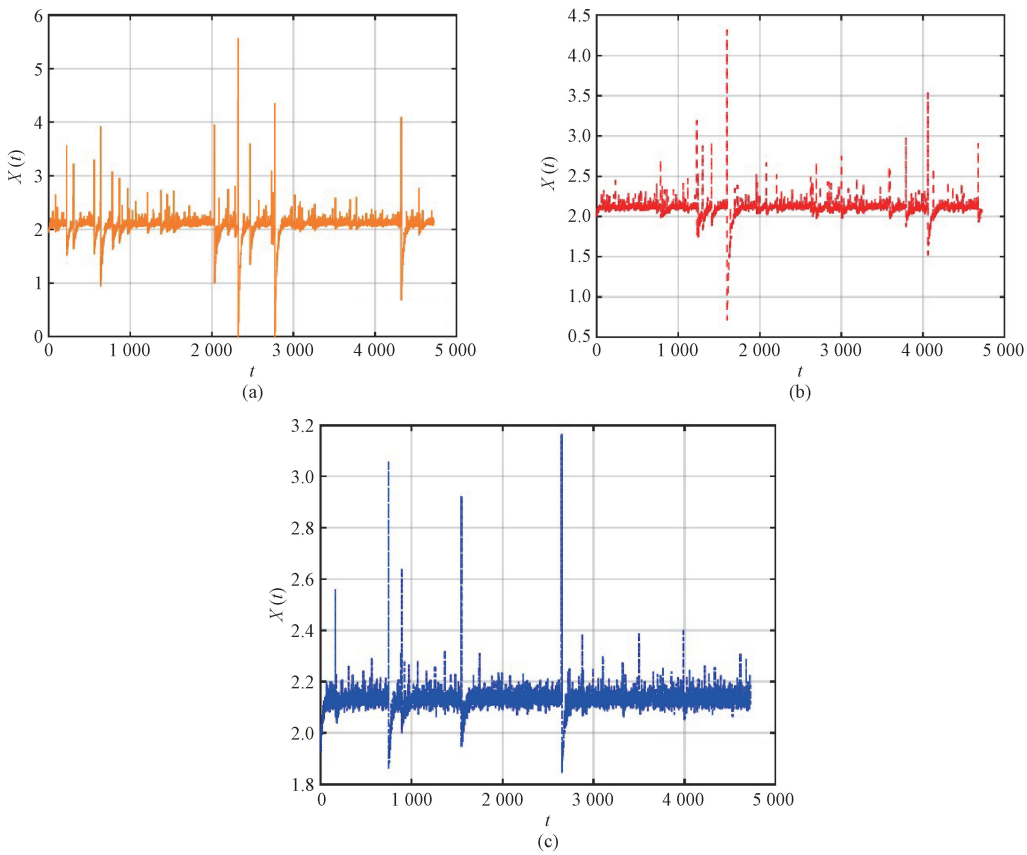


Fig. 5 Solutions of $X(t)$ for different values of α in SCIR model using Shibor data: (a) α = 4/3; (b) α = 3/2; (c) α = 7/4

4 Conclusions

This study focuses on the convergence of long-term interest rates. Unlike most prior studies that rely on the classical CIR model, this paper introduces an SCIR model driven by an α-stable Lévy process, which better captures empirical characteristics such as discontinuities and heavy tails commonly observed in financial markets. The traditional CIR model tends to underperform in such settings, making the SCIR model a more realistic alternative for modeling long-term interest rate dynamics.

We first provide a rigorous theoretical analysis of the convergence properties of the SCIR model, demonstrating

that the long-run time average of the short-term interest rate converges almost surely to a finite limit. To validate the empirical applicability of the model, we use Shibor as our dataset. Shibor, the benchmark rate for short-term interbank lending in China's banking system, effectively reflects market funding costs and is widely used in financial practice.

Through numerical simulations and empirical analysis based on Shibor data, we find that the SCIR model not only captures the mean-reverting behavior of interest rates but also exhibits robustness in the presence of market shocks and non-Gaussian disturbances. This makes the SCIR model a more practical and stable tool for modeling long-term yields and contributes to financial decision-making frameworks.

From a financial management perspective, the SCIR model enhances the accuracy of interest rate forecasts, supporting more effective asset-liability management and risk exposure control. It also offers valuable insights for assessing monetary policy transmission and can assist regulators in understanding market expectations. Furthermore, the model facilitates more reasonable pricing and hedging of interest rate-sensitive instruments such as bonds and swaps, thereby contributing to market stability and efficient resource allocation.

In summary, this research not only enriches the theoretical framework for interest rate modeling but also broadens the application of Lévy processes in financial economics, offering methodological references for future studies.

References

[1] Fu Z F, Li Z H. Stochastic equations of non-negative processes with jumps [J]. *Stochastic Processes and Their Applications*, 2010, 120(3): 306-330.

[2] Li Z H, Mytnik L. Strong solutions for stochastic differential equations with jumps [J]. *Annales de L'Institut Henri Poincaré, Probabilités et Statistiques*, 2011, 47(4): 1055-1067.

[3] Barczy M, Ben Alaya M, Kebaier A, et al. Asymptotic properties of maximum likelihood estimator for the growth rate of a stable CIR process based on continuous time observations [J]. *Statistics*, 2019, 53(3): 533-568.

[4] Bayraktar E, Clément E. Estimation of a pure-jump stable Cox-Ingersoll-Ross process [J]. *Bernoulli*, 2025, 31(1): 484-508.

[5] Jiao Y, Ma C H, Scotti S. Alpha-CIR model with branching processes in sovereign interest rate modeling [J]. *Finance and Stochastics*, 2017, 21(3): 789-813.

[6] Li Z H, Ma C H. Asymptotic properties of estimators in a stable Cox-Ingersoll-Ross model [J]. *Stochastic Processes and Their Applications*, 2015, 125(8): 3196-3233.

[7] Zhang Z Z, Zhang E H, Tong J Y. Necessary and sufficient conditions for ergodicity of CIR model driven by stable processes with Markov switching [J]. *Discrete and Continuous Dynamical Systems: B*, 2018, 23(6): 2433-2455.

[8] Deelstra G, Delbaen F. Long-term returns in stochastic interest rate models [J]. *Insurance: Mathematics and Economics*, 1995, 17(2): 163-169.

[9] Cox J C, Ingersoll Jr J E, Ross S A. A theory of the term structure of interest rates [J]. *Econometrica*, 1985, 53(2): 385-407.

[10] Mandelbrot B. The variation of some other speculative prices [J]. *The Journal of Business*, 1967, 40(4): 393-413.

[11] Fama E F. The behavior of stock-market prices [J]. *The Journal of Business*, 1965, 38(1): 34-105.

[12] Rosinski J, Woyczynski W A. On Itô stochastic integration with respect to *P*-stable motion: inner clock, integrability of sample paths, double and multiple integrals [J]. *The Annals of Probability*, 1986, 14(1): 271-286.

[13] Samorodnitsky G, Taqqu M S. Stable Non-Gaussian random processes: Stochastic models with infinite variance [M]. Boca Raton: CRC, 1994.

[14] Sato K. Lévy processes and infinitely divisible distributions [M]. Cambridge: Cambridge University Press, 1999.

[15] Bertoin J. Lévy processes [M]. Cambridge: Cambridge University Press, 1996.

由 α-稳定 Lévy 过程驱动的 CIR 模型的长期收益

路雯涵，闫理坦*，陶　清
东华大学 数学与统计学院，上海 201620

摘　要：利率在国民经济运行、资本流动、资产定价及保险产品收益中起着至关重要的作用，然而其在非高斯噪声下的长期动态特性仍未得到充分研究。本文通过引入 α-稳定 Lévy 过程对传统科克斯–英格索尔–罗斯（Cox-Ingersoll-Ross，CIR）模型进行扩展，探究在稳定型 CIR（stable CIR，SCIR）模型中，短期利率的长期时间平均值是否收敛于一个有限极限。本文在不同参数设置下进行了数值模拟，以验证模型收敛行为在不同情景下的稳健性，并利用上海银行间同业拆放利率（Shanghai interbank offered rate，Shibor）数据进行了实证分析。结果表明，在适当条件下，SCIR 模型下短期利率的长期时间平均值收敛为一个有限极限，模拟结果与 Shibor 数据均支持这一结论。研究结果为解读非高斯噪声驱动下利率的长期运行规律提供了理论依据，并对金融资产定价与风险管理具有实际意义。

关键词：长期收益；CIR 模型；α-稳定 Lévy 过程；以概率 1 收敛