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Heisenberg Uncertainty Principle for n-Dimensional Linear Canonical Transforms

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Abstract: The uncertainty principle proposed by German physicist Heisenberg in 1927 is a basic principle of quantum mechanics and signal processing. Since linear canonical transformation has been widely used in various fields of signal processing recently and Heisenberg uncertainty principle has been endowed with new expressive meaning in linear canonical transforms domain, in this manuscript, an improved Heisenberg uncertainty principle is obtained in linear canonical transforms domain.

Keywords: Heisenberg uncertainty principle; linear canonical transforms; Pitt inequality

1 Introduction

In 1927, the German physicist Heisenberg proposed the uncertainty principle also called Heisenberg-Weyl inequality. Now the Heisenberg uncertainty principle has been applied to quantum mechanics and signal processing. It means that the position and the momentum of a particle can not be determined simultaneously in a quantum mechanical systems [2]. Equivalently, Heisenberg uncertainty principle can be seen the temporal resolution and the frequency resolution of any signal which can not be infinitely improved simultaneously in signal processing community [3].

In fact, the temporal resolution of a unit signal $f(x)\left(\int_{-\infty}^{+\infty}|f(x)|^2\mathrm{d}x=1\right)$ is defined as [4]

$$\Delta_t^2=\int_{-\infty}^{+\infty}(x-x_0)^2|f(x)|^2\mathrm{d}x \tag{1}$$

and the frequency resolution of $f(x)$ is defined as

$$\Delta_u^2=\int_{-\infty}^{+\infty}(u-u_0)^2\left|\hat{f}(u)\right|^2\mathrm{d}u \tag{2}$$

where $\hat{f}(u)=\int_{-\infty}^{+\infty}f(x)\mathrm{e}^{-\mathrm{i}2\pi xu}\mathrm{d}x$ (Fourier transform (FT) of $f(x)$). Then the inequality

$$\Delta_t^2\Delta_u^2\geqslant\frac{1}{16\pi^2} \tag{3}$$

The so-called Heisenberg-Weyl inequality can be obtained by using Cauchy inequality. In 1995, a sharp form of Pitt inequality is obtained by Beckner [5], and then logarithmic uncertainty principle

$$\int_{-\infty}^{+\infty}\ln|u|\left|\hat{f}(u)\right|^2\mathrm{d}u+\int_{-\infty}^{+\infty}\ln|x|\,|f(x)|^2\mathrm{d}x\geqslant\frac{\Gamma'(1/4)}{\Gamma(1/4)}-\ln\pi \tag{4}$$

can be gotten for any unit $f(x)$ belonged to Schwartz space [6].

Heisenberg uncertainty principle and logarithmic uncertainty principle are associated with Fourier transform (FT) and there are some general results such as logarithmic uncertainty principle in linear canonical transforms (LCT) domain [7]. LCT, firstly introduced by Collins in 1970, is a generalization of FT and can be used in differential equations, optical systems analysis [8]. With the rapid development of fractional

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Fourier transform, LCT has been paid more and more attention in applied mathematics and signal processing community such as the filtering theory [9], the frame theory [10], the sampling theory [11] and the discrete algorithms [12]. Furthermore, LCT can be used to radar signal processing, communication signal processing, the image encryption, denoising and so on.

Since LCT is very useful, it is worth to study Heisenberg uncertainty principle in LCT domain. Hence, in this manuscript, we will improve Heisenberg uncertainty principle in LCT domain and n-dimensional Heisenberg uncertainty principle also be obtained. The manuscript is organized as follows. Section 2 introduces some general definitions and gives some results. In Section 3, we calculate n-dimensional Heisenberg uncertainty principle in LCT domain.

2 Preliminaries

Before we proceed, some important definitions and results related to the LCT and Heisenberg uncertainty principle are reviewed in this section.

2.1 Linear Canonical Transforms

We frequently use the one-dimensional LCT in signal processing [1] as

$$F_{a,b,c,d}(u)=\begin{cases}\dfrac{1}{\sqrt{2\pi b\mathrm{i}}}\displaystyle\int_{-\infty}^{+\infty}f(t)\mathrm{e}^{\frac{\mathrm{i}(at^2-2ut+du^2)}{2b}}\mathrm{d}t, & b\neq 0\\ \sqrt{d}\mathrm{e}^{\mathrm{i}\frac{cdu^2}{2}}f(du), & b=0\end{cases} \tag{5}$$

where the parameters satisfy $ad-bc=1$. While the inverse transform of the one-dimensional LCT (ILCT) is given by the following formula

$$f(t)=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi b\mathrm{i}}}F_{a,b,c,d}(u)\mathrm{e}^{\frac{\mathrm{i}(-at^2+2ut-du^2)}{2b}}\mathrm{d}u \tag{6}$$

Generally, for each $2n\times 2n$ symplectic matrix $\boldsymbol{M}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}\left(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{M}=\boldsymbol{J},\ \boldsymbol{J}=\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}_n\\ -\boldsymbol{I}_n & \boldsymbol{0}\end{pmatrix}\right)$, $\boldsymbol{M}^{\mathrm{T}}$ is conjugate transpose of $\boldsymbol{M}$. n-dimensional LCT [13] is defined as

$$F_{\boldsymbol{M}}(\boldsymbol{u})=\int_{\Re^n}C(\boldsymbol{M})(\boldsymbol{u},\boldsymbol{x})f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{7}$$

where

$$C(\boldsymbol{M})(\boldsymbol{u},\boldsymbol{x})=\frac{\mathrm{e}^{-\mathrm{i}n\pi/4}}{\sqrt{\det(\boldsymbol{B})2^n\pi^n}}\mathrm{e}^{\mathrm{i}\left(\frac{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{u}-2\boldsymbol{u}^{\mathrm{T}}\left(\boldsymbol{B}^{\mathrm{T}}\right)^{-1}\boldsymbol{x}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{x}}{2}\right)}$$

For more detailed definitions and properties of the LCT, one can refer to [1].

2.2 Heisenberg Uncertainty Principle for Complex Signal

In this subsection, we recall Heisenberg uncertainty principle for complex signal. Supposing a unit signal $f(x)=\rho(x)\mathrm{e}^{\mathrm{i}\varphi(x)}$, we have

$$\begin{aligned}\Delta_t^2\Delta_u^2=&\int_{-\infty}^{+\infty}|(x-x_0)\,\rho(x)|^2\mathrm{d}x\cdot\int_{-\infty}^{+\infty}\left|(u-u_0)\,\hat{f}(u)\right|^2\mathrm{d}u=\\ &\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}|(x-x_0)\,\rho(x)|^2\mathrm{d}x\cdot\\ &\int_{-\infty}^{+\infty}\left|(\rho'(x)+\mathrm{i}\varphi'(x))\mathrm{e}^{\mathrm{i}\varphi(x)}\right|^2\mathrm{d}u\geqslant\\ &\frac{1}{4\pi^2}\left|\int_{-\infty}^{+\infty}(x-x_0)\,\rho(x)(\rho'(x)+\mathrm{i}\varphi'(x))\mathrm{d}x\right|^2=\\ &\frac{1}{4\pi^2}+\frac{1}{4\pi^2}\left|\int_{-\infty}^{+\infty}(x-x_0)\,\rho(x)\varphi'(x)\mathrm{d}x\right|^2\end{aligned} \tag{8}$$

Using this result, Heisenberg uncertainty principle in LCT domain and some other uncertainty principle can be improved [14].

2.3 The Sharp Pitt Inequality

Since logarithmic uncertainty principle is obtained from the sharp Pitt inequality, in this subsection, the next lemma is needed [5].

Lemma 1 Let $f(x)$ be a rapidly decreasing function [6] and $0\leqslant\alpha\leqslant 1$, then

$$\int_{\Re^n}|u|^{-\alpha}\left|\hat{f}(u)\right|^2\mathrm{d}u\leqslant\frac{\pi^{\alpha}\Gamma^2((n-\alpha)/4)}{\Gamma^2((n+\alpha)/4)}\int_{\Re^n}|x|^{\alpha}|f(x)|^2\mathrm{d}x \tag{9}$$

where $\Gamma(x)$ is the Gamma function. In fact, Lemma 1 can be obtained from Stein-Weiss fractional integral inequality [5].

3 Main Results

In this section, let $f(x)$ be a unit signal with $2\leqslant\alpha$, then Rassias obtained the following in-

equality by using the Hölder inequality and the Plancherel-Parseval-Rayleigh identity [15]

$$\Delta_t^\alpha \Delta_u^\alpha \geqslant \frac{1}{(4\pi)^\alpha}$$

where $\Delta_t^\alpha = \int_{-\infty}^{+\infty} (x-x_0)^\alpha |f(x)|^2 \mathrm{d}x$ and $\Delta_u^\alpha = \int_{-\infty}^{+\infty} (u-u_0)^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u \Delta_t^2 \Delta_u^2 \geqslant \frac{1}{4}$. Especially, when $\alpha = 2$, it is just Heisenberg uncertainty principle. In fact, this result can be improved to get the following lemma.

Lemma 2 Supposing $f(x)$ is a rapidly decreasing function and $2 \leqslant \alpha \leqslant 3$, we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u \geqslant \frac{C_\alpha}{16\pi^2} \quad (10)$$

where $C_\alpha = \dfrac{\Gamma^2((1+\alpha-2)/4)}{\pi^{\alpha\cdot 2}\Gamma^2((1-\alpha+2)/4)}$.

Proof Let $g(x) = xf(x)$, then, we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} |x|^{\alpha-2} |g(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u$$

By using Lemma 1, we get

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u \geqslant \frac{C_\alpha}{4\pi^2} \int_{-\infty}^{+\infty} |u|^{2-\alpha} |\hat{g}(u)|^2 \mathrm{d}u \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u$$

Then, by using Cauchy inequality, we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u \geqslant \frac{C_\alpha}{4\pi^2} \left(\int_{-\infty}^{+\infty} |\hat{g}(u)| \left|u\hat{f}(u)\right| \mathrm{d}u \right)^2 \geqslant \frac{C_\alpha}{4\pi^2} \left(\left| \int_{-\infty}^{+\infty} \hat{g}(u)\hat{f}'(u) \mathrm{d}u \right| \right)^2$$

From Moyal identical equations, we obtain

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \left|\hat{f}(u)\right|^2 \mathrm{d}u \geqslant \frac{C_\alpha}{16\pi^2} \left(\left| \int_{-\infty}^{+\infty} g(x) f'(x) \mathrm{d}x \right| \right)^2 = \frac{C_\alpha}{16\pi^2}$$

This has finished the proof.

Associated with LCT, the following theorem can be obtained.

Theorem 1 Let $f(x)$ be a rapidly decreasing function and $2 \leqslant \alpha \leqslant 3$, then we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha |F_{a,b,c,d}(u)|^2 \mathrm{d}u \geqslant |b|^\alpha 2^{\alpha-4} \frac{\Gamma^2((1+\alpha-2)/4)}{\Gamma^2((1-\alpha+2)/4)} \quad (11)$$

Proof Let $g(x) = f(x)\mathrm{e}^{\frac{ax^2}{2b}\mathrm{i}}$, then, we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha |F_{a,b,c,d}(u)|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} |x|^\alpha |g(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \cdot \left| \frac{1}{\sqrt{2\pi b\mathrm{i}}} \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{\frac{\mathrm{i}(at^2-2ut+du^2)}{2b}} \mathrm{d}t \right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} |x|^\alpha |g(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha \cdot \left| \frac{1}{\sqrt{2\pi b\mathrm{i}}} \int_{-\infty}^{+\infty} g(t) \mathrm{e}^{\frac{-ut}{b}\mathrm{i}} \mathrm{d}u \right|^2 \mathrm{d}u$$

Setting $v = \dfrac{u}{2\pi b}$, we have

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha |F_{a,b,c,d}(u)|^2 \mathrm{d}u = (2\pi|b|)^{\alpha-2} \int_{-\infty}^{+\infty} |x|^\alpha |g(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |v|^\alpha \left| \int_{-\infty}^{+\infty} g(t) \mathrm{e}^{-2\pi vt\mathrm{i}} \mathrm{d}t \right|^2 \mathrm{d}v$$

By using Lemma 2, we get

$$\int_{-\infty}^{+\infty} |x|^\alpha |f(x)|^2 \mathrm{d}x \int_{-\infty}^{+\infty} |u|^\alpha |F_{a,b,c,d}(u)|^2 \mathrm{d}u \geqslant (2\pi|b|)^\alpha \frac{C_\alpha}{16\pi^2} = |b|^\alpha 2^{\alpha-4} \frac{\Gamma^2((1+\alpha-2)/4)}{\Gamma^2((1-\alpha+2)/4)}$$

In fact, Theorem 1 is just Heisenberg uncertainty principle in LCT domain when $\alpha = 2$. Next, the following theorem can be obtained associated with the definition of n-dimensional LCT.

Theorem 2 Let $f(x)$ be a rapidly decreasing function and $2 \leqslant \alpha \leqslant 3$, then we have

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}\geqslant \left(\sum_{k=1}^{n}\lambda_k^{\frac{\alpha}{4}}\right)^2 2^{\alpha-4}\frac{\Gamma^2((1+\alpha-2)/4)}{\Gamma^2((1-\alpha+2)/4)} \tag{12}$$

Proof Let $g(\boldsymbol{x})=f(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{x}}{2}}$, then we have

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}=$$

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|g(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}\left|\int_{\Re^n}\frac{g(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\boldsymbol{u}^{\mathrm{T}}\left(\boldsymbol{B}^{\mathrm{T}}\right)^{-1}\boldsymbol{x}}}{\sqrt{\det(\boldsymbol{B})2^n\pi^n}}\mathrm{d}\boldsymbol{x}\right|^2\mathrm{d}\boldsymbol{u}$$

Setting $\boldsymbol{v}=(\boldsymbol{B}^{\mathrm{T}})^{-1}\boldsymbol{x}$ and $h(\boldsymbol{v})=g(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{v})$, we have

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}=$$

$$\int_{\Re^n}|v^{\mathrm{T}}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}v|^{\frac{\alpha}{2}}|h(v)|^2\mathrm{d}v\int_{\Re^n}|\boldsymbol{u}|^{\alpha}\left|\int_{\Re^n}\frac{h(v)\mathrm{e}^{\mathrm{i}\boldsymbol{u}^{\mathrm{T}}v}}{\sqrt{2^n\pi^n}}\mathrm{d}v\right|^2\mathrm{d}\boldsymbol{u}$$

Then, when $\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}=\boldsymbol{P}^{\mathrm{T}}\Lambda\boldsymbol{P}$, we can have by setting $\boldsymbol{w}=\boldsymbol{P}\boldsymbol{v}$ and $\boldsymbol{t}=\boldsymbol{P}\boldsymbol{u}$

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}=$$

$$\int_{\Re^n}\sum_{k=1}^{n}|\lambda_k w_k^2|^{\frac{\alpha}{2}}\left|\overline{h}(\boldsymbol{w})\right|^2\mathrm{d}\boldsymbol{w}\int_{\Re^n}|\boldsymbol{t}|^{\alpha}\left|\int_{\Re^n}\frac{\overline{h}(\boldsymbol{w})\mathrm{e}^{\mathrm{i}t^{\mathrm{T}}\boldsymbol{w}}}{\sqrt{2^n\pi^n}}\mathrm{d}\boldsymbol{w}\right|^2\mathrm{d}t$$

where λ_k are the eigenvalues of $\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}$ and $\overline{h}(\boldsymbol{w})=h(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{w})$. Using Lemma 2 and the inequality $\left(\sum_{k=1}^{n}|\boldsymbol{x}_k|^2\right)^{\frac{\alpha}{2}}\geqslant\sum_{k=1}^{n}|\boldsymbol{x}_k|^{\alpha}$, we obtain

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}\geqslant$$

$$\int_{\Re^n}\sum_{k=1}^{n}\lambda_k^{\frac{\alpha}{2}}|\boldsymbol{w}_k|^{\alpha}\left|\overline{h}(\boldsymbol{w})\right|^2\mathrm{d}\boldsymbol{w}\cdot$$

$$\int_{\Re^n}\sum_{k=1}^{n}|\boldsymbol{t}_k|^{\alpha}\left|\int_{\Re^n}\frac{\overline{h}(\boldsymbol{w})\mathrm{e}^{\mathrm{i}\boldsymbol{t}^{\mathrm{T}}\boldsymbol{w}}}{\sqrt{2^n\pi^n}}\mathrm{d}\boldsymbol{w}\right|^2\mathrm{d}t\geqslant$$

$$\left[\sum_{k=1}^{n}\left(\int_{\Re^n}\lambda_k^{\frac{\alpha}{2}}|\boldsymbol{w}_k|^{\alpha}\left|\overline{h}(\boldsymbol{w})\right|^2\mathrm{d}\boldsymbol{w}\cdot\right.\right.$$

$$\left.\left.\int_{\Re^n}|t_k|^{\alpha}\left|\int_{\Re^n}\frac{\overline{h}(\boldsymbol{w})\mathrm{e}^{\mathrm{i}t^{\mathrm{T}}\boldsymbol{w}}}{\sqrt{2^n\pi^n}}\mathrm{d}\boldsymbol{w}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\right]^2$$

Thus, from Theorem 1, we have

$$\int_{\Re^n}|\boldsymbol{x}|^{\alpha}|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\Re^n}|\boldsymbol{u}|^{\alpha}|F_M(\boldsymbol{u})|^2\mathrm{d}\boldsymbol{u}\geqslant \left(\sum_{k=1}^{n}\lambda_k^{\frac{\alpha}{4}}\right)^2 2^{\alpha-4}\frac{\Gamma^2((1+\alpha-2)/4)}{\Gamma^2((1-\alpha+2)/4)}$$

In fact, Theorem 2 is a generalization of Theorem 5 in [16]. In particularly, Heisenberg uncertainty principle is just Theorem 2 when $\alpha=2$ and $n=1$.

4 Conclusion

In this manuscript, we obtain a Heisenberg uncertainty principle in LCT domain by using sharp Pitt inequality and this result improves the theorem of Rassias. In fact, Theorem 2 can be obtained only using the integral transform, sharp Pitt inequality and Cauchy. In the following work, we find Pitt inequality holds only for $0\leqslant\alpha\leqslant 1$, hence, it is worth to discuss Pitt inequality for $1\leqslant\alpha\leqslant 2$.

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