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Adaptive Short-Time Fractional Fourier Transform Based on Minimum Information Entropy

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Abstract: Traditional short-time fractional Fourier transform (STFrFT) has a single and fixed window function, which can not be adjusted adaptively according to the characteristics of frequency and frequency change rate. In order to overcome the shortcomings, the STFrFT method with adaptive window function is proposed. In this method, the window function of STFrFT is adaptively adjusted by establishing a library containing multiple window functions and taking the minimum information entropy as the criterion, so as to obtain a time-frequency distribution that better matches the desired signal. This method takes into account the time-frequency resolution characteristics of STFrFT and the excellent characteristics of adaptive adjustment to window function, improves the time-frequency aggregation on the basis of eliminating cross term interference, and provides a new tool for improving the time-frequency analysis ability of complex modulated signals.

Keywords: short-time fractional Fourier transform (STFrFT); adaptive algorithm; minimum information entropy

1 Introduction

Since the 1940s, many famous scholars began to study the representation method of time-frequency characteristics. After 80 years of development, time-frequency analysis technology has become an important branch of signal processing technology. According to the different kernel functions and optimization methods of time-frequency analysis, it can be divided into linear time-frequency analysis method, bilinear time-frequency analysis method, rotary time-frequency analysis method and adaptive time-frequency analysis method [1]. The rotating time-frequency analysis method mainly depends on rotating the time-frequency plane to achieve the effect of signal analysis. The representative of this class is the fractional Fourier transform [2-4]. By combining the characteristics of short-time Fourier transform and fractional Fourier transform, the short-time fractional Fourier transform (STFrFT) has been obtained [5,6]. It greatly improves the ability of local analysis, expands the application scope of fractional Fourier transform, and provides an effective technical way for the detection and estimation of nonlinear FM signal. For example, it is applied to radar detection of human activity and achieves enhanced recogonition performance [7]. However, the short-time window used in STFrFT is often single and fixed, and its time-frequency resolution is fixed. According to the viewpoint of [8], there is no universally applicable window function, which can make different signals obtain high energy aggregation in the time-frequency representation of STFrFT. This makes it possible to produce ambiguity in time or frequency domain when analyzing the unknown signal, which leads to the error of signal feature analysis. Since a single short-

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time window can not meet the requirements of optimal time-frequency analysis for the whole duration of the signal, it is necessary to adjust the window function flexibly according to the time-varying characteristics of the signal in order to ensure the quality of time-frequency analysis.

2 Adaptive Version of STFrFT

According to Heisenberg uncertainty criterion, high time resolution corresponds to low frequency resolution, while high frequency resolution corresponds to low time resolution. Because the STFrFT is based on the fixed local segmentation of the signal in the time domain, and then the quasi linear method is used to calculate it, the selection of the optimal window width of STFrFT becomes extremely complex for a nonlinear frequency-modulated signal whose frequency modulation rate changes with time. In other words, it is difficult to select the window function which is suitable for both low-frequency and high-frequency change rate segments for the signal whose frequency change is too fast. As a result, the window function of STFrFT can only make the signal achieve the best time-frequency resolution in a certain interval, while it is passive "accommodation" in other intervals. It is very difficult to determine an optimal window function that exactly matches the whole signal period, so that the signal can obtain high frequency resolution in the whole time interval.

In order to obtain better time-frequency resolution effect, we can consider adding a specific algorithm in the STFrFT operation, such as adjusting the window function, adjusting the operation method or selecting the appropriate algorithm adaptively according to the prior information obtained from the signal to be measured in advance.

Fig. 1 shows the adaptive STFrFT method which uses different window functions to form a window function library for adaptive adjustment. By adopting certain adaptive criteria, different window functions are used for operation at different sampling times. At each instant, the short-time window function and window width can be adaptively adjusted according to the signal change law, so that the signal intercepted in the short-time window at each moment can meet the "quasi stationary" state, that is, the signal can be processed according to the standard of approximate LFM signal.

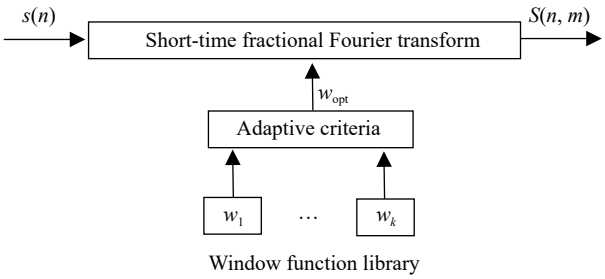

Fig. 1 Block diagram of adaptive STFrFT

Adaptive STFrFT requires that the window function and window width used at each sampling time should match the signal characteristics at that time. Long time window is used to improve the frequency resolution at the time when the frequency changes slowly, and short time window is used to improve the time resolution at the time when the frequency changes fast. In order to make up for the defect of low time-frequency resolution caused by the mismatch between window function and signal, the trade-off between time resolution and frequency resolution is achieved. When the time-frequency resolution is improved, the accuracy of instantaneous frequency estimation for the whole signal will be improved accordingly. Therefore, to achieve the above adaptive requirements, it is necessary to seek the identification information that can match the signal characteristics.

3 Information Entropy

The research of time-frequency analysis for the signal needs to be based on its measurable information. That is to say, to study the problem

of information transmission and processing in theory, we must give a quantitative description of the information. So the concept of information entropy is put forward. Shannon, an American scientist, first studied signal processing by using entropy. In 1948, he applied the concept of thermal entropy in thermodynamics to the field of signal processing, and defined information entropy as the average amount of information output [9]. Information entropy is an important concept to measure information uncertainty in information theory. The greater the uncertainty of variables, the greater the entropy. On the contrary, the more certain the value of variables, the smaller the entropy. Therefore, the entropy value can be used to judge the uncertainty degree of the time-frequency resolution unit in the time-frequency distribution.

3.1 Definition

Let a discrete probability distribution sequence be $p=(p_1,p_2,\cdots,p_n)$, then according to the definition of information entropy, its information entropy can be written as $H(p)=-\sum_i p_i \ln p_i$, where $i=1,2\cdots,n$ represents all possible values of the output random variable, and p_i denotes the probability of each value. By analogy, in the time-frequency distribution of signal, if the frequency distribution of signal at a certain time, $p=(p_1, p_2,\cdots,p_F)$, is regarded as a probability distribution sequence, then the frequency probability distribution of signal at time t can be obtained as

$$p_t(f)=\frac{|X(t,f)|^2}{\int_{-\infty}^{\infty}|X(t,f)|^2\mathrm{d}f} \qquad (1)$$

where $X(t,f)$ is the energy size of the signal at the spot (t,f). Then at time t, the information entropy of the frequency contained in the signal can be written as

$$H_t(p_t)=-\sum_F p_t \ln p_t \qquad (2)$$

$H_t(p_t)$ is called entropy function, which is used to measure and express the information entropy. F is the number of frequency resolution units, and its value is related to the time length of the signal and the time-frequency analysis algorithm.

3.2 Basic Properties

Entropy functions, as a special class of functions, usually have some of the following properties.

3.2.1 Symmetry

When the order of the variables in the probability distribution sequence is changed arbitrarily, the value of the entropy function remains unchanged, i.e.

$$H_t(p_1,p_2,\cdots,p_F)=\cdots=H_t(p_F,p_1,\cdots p_2,) \qquad (3)$$

This property shows that information entropy is only related to the overall structure of random variables. In the time-frequency distribution, it is related to the overall statistical characteristics of the frequency resolution unit. If the statistical characteristics of two frequency resolution units are the same, then their information entropy is the same.

3.2.2 Extremum

$$0=H_t(1,0,\cdots,0)\le H_t(p_1,p_2,\cdots,p_F)\le H_t(1/q,1/q,\cdots,1/q)=\log q \qquad (4)$$

This property shows that in the discrete case, for the time-frequency distribution with q frequency resolution units, the information entropy reaches the maximum when the energy distribution of frequency resolution units is uniform and the probability of occurrence is possible. When only one frequency unit has the maximum energy and the other frequency resolution units have no energy distribution, the information entropy reaches the minimum. This is also an important basis for the algorithm to track the energy peak by using the minimum information entropy.

4 Adaptive STFrFT Based on Minimum Information Entropy

4.1 Adaptive Algorithm

$w=\{w_1,w_2,w_3,\cdots,w_M\}$ is defined as a series of

window functions with different parameters, among which $\|w_i\|^2 = 1,\ i = 1, 2, \cdots, M$. Then w becomes the window function library for adaptive fractional Fourier transform, and each element in the library becomes the window function that can be selected for STFrFT. Then the expression of adaptive STFrFT for input signal $x(t)$ can be written as

$$X_{\mathrm{ASTFrFT},w_{ru}}(t,u) = \int_{-\infty}^{\infty} x(\tau)\, w_{ru}(\tau - t) K_{\alpha}(t,u)\, \mathrm{d}\tau \tag{5}$$

It can be found from (5) that the algorithm of the adaptive STFrFT is the same as that of the STFrFT except that the window function w_{ru} is adaptively adjusted. The difference is that the adaptive STFrFT adaptively selects the best window function from the window function library at each time. It can make the signal achieve the best resolution at every time, so as to obtain the best resolution in the whole time and frequency domain.

4.2 Adaptive Criteria

At present, there are many research directions and entry points about adaptive criteria in domestic and foreign literatures, and there are also many adaptive criteria that can be used for adaptive STFrFT. Among them, the common methods that are suitable for STFrFT are spectral kurtosis [10], linear chirp local approximation [11], maximum correlation method [12], Wigner distribution-based method [13], chirp rate [14] and minimum entropy method [15,16]. When using the maximum correlation criterion for adaptive short-time Fourier transform, its operation has all the good characteristics of maximum likelihood estimation, which can effectively reduce the amount of computation of adaptive transform, but its accuracy for instantaneous frequency estimation is not ideal. When using the Wigner distribution-based criterion for adaptive STFrFT, although the operation accuracy is improved in the case of high signal-to-noise ratio, bilinear time-frequency analysis tools such as Wigner distribution are very sensitive to noise. According to the good anti-noise performance of fractional Fourier transform, the minimum information entropy criterion is suitable to track the peak value of time-frequency energy when using adaptive STFrFT for time-frequency analysis. Therefore, this paper focuses on the performance of adaptive STFrFT based on minimum information entropy criterion.

The time-frequency distribution from the adaptive STFrFT of a signal can be understood as a probability distribution sequence. And the probability distribution of frequency resolution unit at time t can defined as

$$p_t(u) = \frac{|X_{\mathrm{STFrFT},w}(t,u)|^2}{\int_{-\infty}^{\infty} |X_{\mathrm{STFrFT},w}(t,u)|^2 \mathrm{d}u} \tag{6}$$

where $X_{\mathrm{STFrFT},w}(t,u)$ is the STFrFT of input signal, $x(t)$, with the window function w. Thus, the information entropy can be expressed as

$$H_t(p_t) = -\int_{-\infty}^{\infty} p_t(u) \cdot \ln p_t(u) \mathrm{d}u \tag{7}$$

For the convenience of calculation, (6) and (7) can be converted into their discrete version as

$$p_n(k) = \frac{|X_{\mathrm{STFrFT},w}(n,k)|^2}{\sum_{k=1}^{N} |X_{\mathrm{STFrFT},w}(n,k)|^2} \tag{8}$$

$$H_n = -\sum_{k=1}^{N} p_n(k) \cdot \ln p_n(k) \tag{9}$$

In this case, H_n is the information entropy of frequency distribution at time t after adaptive STFrFT. Since the size of H_n can reflect the energy aggregation of time-frequency distribution of the signal, by searching for the minimum information entropy, the optimal window width can be obtained to achieve the highest energy aggregation at any time. Thus, in order to obtain the best time-frequency aggregation, the criterion about minimum information entropy used for adaptive STFrFT can be expressed as

$$w_{ru} = \arg \min_{1 \leqslant m \leqslant M} H_n(w_m) \tag{10}$$

where $H_n(w_m)$ is the information entropy corresponding to the mth element of the window function library with M elements. (10) is the adaptive criterion about minimum information entropy.

4.3 Window Function Library

The selection of window function library w depends on the characteristics of the signal, $x(t)$, to be measured. A good adaptive criterion can greatly improve the time-frequency analysis accuracy of adaptive STFrFT. Under the adaptive criterion, in order to reduce the computation complexity, it is often expected that the elements in the window function library should be as few as possible. However we also want to enrich the elements in the library as much as possible in order to meet the efficiency of time-frequency analysis. In addition, when the signal-to-noise ratio of the signal to be measured decreases, the energy generated by the noise may submerge the energy of the desired signal. In this case, if there are too many mismatched window functions with the desired signal in the library, such window functions may match the noise more, which will reduce the accuracy of time-frequency analysis.

Therefore, in practical application, the window function library should be chosen according to the characteristics of the signal to be measured. Under the condition that the characteristics of the signal to be measured are unknown and lack of prior information, a library with many kinds of window functions can be used to analyze the intercepted signal. Then the window function signal with poor resolution is removed by mathematical statistics method, and the window function signal with good resolution is retained. Finally, the adjusted window function library is used for STFrFT.

In conclusion, when the signal to be measured is unknown, it is necessary to use window function library which matches different signals. Since the Gaussian window function has the best time-frequency resolution effect [17], in this paper, the Gaussian window is still used to form the window function library. When part of the prior information of the signal to be measured is known, the window function matching the signal is used, and different window widths are changed to form a window function library. By searching and matching the signals with different window widths, the window width is adaptively adjusted to achieve the adaptive effect.

Combining the criterion about minimum information entropy and the selection standard of window function library, the basic algorithm steps of adaptive STFrFT based on minimum information entropy are as follows:

Step 1 If the prior information of the signal is obtained, the second step is directly carried out, otherwise, the window function library, w, containing different window function types is constructed. Then each element of w is used to perform the STFrFT of the signal with a small duration, and the corresponding entropy value is obtained. According to the principle of minimum information entropy, obtain the best window function type, and construct the window function library w.

Step 2 If the prior knowledge is obtained in the first step, it is used to construct the window function library, w, with different window widths, and then the signal to be measured is transformed by the STFrFT using w; Otherwise, the window function library, w, constructed in the first step is directly used for the STFrFT; And the corresponding entropy value is obtained.

Step 3 Comparing the information entropy calculated by different window widths at the same time, the window function corresponding to the minimum information entropy is the best window function at that time, and the frequency distribution under the window function is the frequency distribution at that time.

The principle block diagram is shown in Fig. 2.

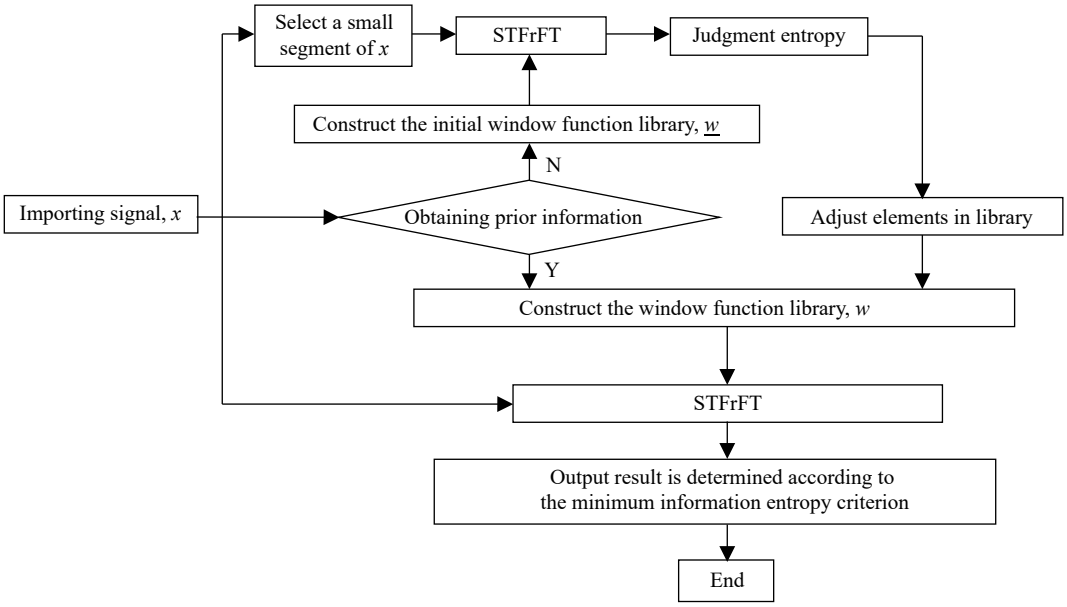

Fig. 2 Functional block diagram of adaptive STFrFT with minmun minimum entropy

5 Simulation Analysis

5.1 Time Frequency Resolution

The third phase nonlinear FM signal is used as the signal to be measured, and its expression is as follows

$$x(t) = \exp\left(\mathrm{j}2\pi\left(f_0 t + 170t^3 - 100t^2\right)\right) \qquad (11)$$

where the initial frequency of the signal is f_0= 30 Hz. And the sampling duration is 1 s, and the sampling frequency is 1 024 Hz. It is assumed that prior information such as the type of signal is unknown. Firstly, a window function library including Gaussian window, rectangular window, Hamming window and Hanning window is set up. Three window widths of 64 points, 128 points and 256 points are selected to perform STFrFT, and the search step is 0.001. Through the minimum information entropy criterion, it can be found that the information entropy of STFrFT is the minimum when Gaussian window is used. After determining the types of window functions, the window function library is simplified again. The new window function library uses Gaussian windows, including 8 points, 16 points, 32 points, 64 points, 100 points, 128 points, 200 points and 256 points. Then, the signal is processed by sliding calculation point by point under different window lengths. The information entropy is calculated according to the instantaneous energy distribution. Finally, the time-frequency distribution is obtained according to the criterion of minimum information entropy, and the results are shown in Fig. 3.

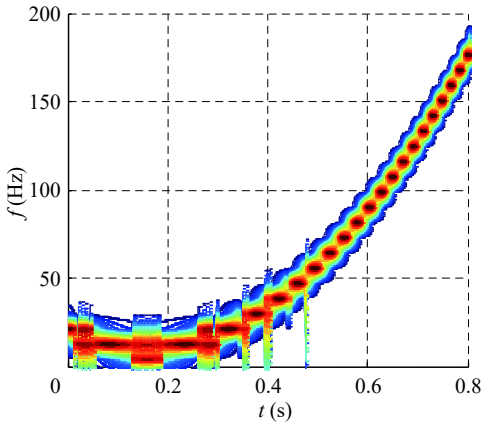

Fig. 3 Time-frequency representation of adaptive STFRFT

As can be seen from Fig. 3, under the adaptive window function library, the window width of STFrFT can be adjusted adaptively, so as to improve the overall time-frequency resolution. When the frequency changes slowly, such as 0.1-0.3 s, the window width is larger, so that the signal can maintain high resolution in frequency domain. When the frequency changes quickly, such as 0.6-0.8 s, the window width is smaller, so as to

maintain good resolution in time domain. Through adaptive adjustment, the adaptive STFrFT has relatively high time-frequency resolution in the whole signal duration.

In time-frequency analysis, the instantaneous frequency estimation of the signal is usually realized by tracking the energy peak in the time-frequency distribution of the signal. When the minimum information entropy criterion is used for adaptive operation, the energy of each frequency resolution unit is also different at each discrete time point and under different window functions. The difference of energy distribution corresponding to the resolution unit with the minimum information entropy is the most obvious, and it is most likely to correspond to the frequency of the signal to be measured. Therefore, taking it as the instantaneous frequency estimation value at that time is the most likely to estimate accurately. Next, the simulation experiment of instantaneous frequency estimation is carried out. All parameters of the signal to be measured are the same as before. The signal to be measured is subjected to STFrFT and adaptive STFrFT respectively, where w=128 is used in STFrFT and $\boldsymbol{w}$=[8, 16, 32, 64, 100, 128, 200, 256] in the adaptive STFrFT. The estimation results are shown in Tab. 1. It can be seen from Tab. 1 that for the signal with the same parameters and conditions, using adaptive STFrFT can obtain higher estimation accuracy in both the period with small frequency change rate and the period with large frequency change rate.

Tab. 1 Accuracy comparison between STFrFT and ASFrFT for estimating instantaneous frequency

Instantaneous time (s)	Instantaneous frequency (Hz)	Estimated frequency (Hz)		Error (Hz)	
		STFrFT	ASFrFT	STFrFT	ASFrFT
0.2	10.4	11.9	11.0	1.5	0.6
0.5	57.3	52.5	55.0	4.8	2.3
0.8	196.1	197.6	195.5	0.7	0.3

5.2 Computational Complexity

In this section, the window function library, w, used for adaptive transformation is divided into two cases: the first $\boldsymbol{w}_1$=[8, 16, 32, 64, 100, 128, 200, 256] and the second $\boldsymbol{w}_2$=[4, 8, 16, 32, 64, 100, 128, 200]. In the initial library, Gaussian window and rectangular window are included, and each window type contains the same number of window widths with $\boldsymbol{w}_1$, totals 16 elements. The computer used is assembly computer, the operating system is Windows XP, the processor is Intel Pentium I3 processor, the memory is 256 MB. Adaptive STFrFT is performed under unknown and known prior conditions respectively, and the operation time consumed is shown in Tab. 2.

Tab. 2 Operation time of adaptive STFrFT

STFrFT		Operation time (s)
Adaptive window	$\boldsymbol{w}_1$ is used	344.6
	$\boldsymbol{w}_2$ is used	366.6
	No prior information	440.1
Fixed window	w=100	37.3

From Tab. 2, it can be seen that the adaptive STFrFT has a large amount of computation, which rooted in the complexity of adaptive STFrFT. The number of sliding operations will be different due to the different selection of window function library, but the difference is not obvious. When the prior information is insufficient, it is necessary to adjust the initial window function library, so it takes the most time in the calculation. It is obvious that the time-frequency resolution of adaptive short-time fractional Fourier transform is obtained by sacrificing the amount of computation. In order to solve this problem, we can make up for it by reducing the search precision and reducing the redundant elements in the window function library.

6 Conclusion

Based on the principles of statistics and information theory, this paper analyzes the adaptive STFrFT using the principle of minimum information entropy. Firstly, the basic concept and prop-

erties of information entropy are introduced, and the concept of minimum information entropy is introduced. Then, aiming at the shortcoming that the short-time window of STFrFT is fixed and single, which is not suitable for processing nonlinear FM signals with complex modulation frequency changes, the adaptive adjustment method of variable window length is applied to STFrFT by taking the improved window function as the breakthrough point, an adaptive STFrFT based on minimum information entropy is proposed. This transform adaptively adjusts the window width according to the signal characteristics to obtain the time-frequency distribution results that adapt to the changes of time-frequency characteristics. Finally, simulation results show the effectiveness of the proposed adaptive STFrFT, but it also costs more computational complexity.

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