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A New Tensor Factorization Based on the Discrete Simplified Fractional Fourier Transform

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Abstract: Tensor analysis approaches are of great importance in various fields such as computation vision and signal processing. Thereinto, the definitions of tensor-tensor product (t-product) and tensor singular value decomposition (t-SVD) are significant in practice. This work presents new t-product and t-SVD definitions based on the discrete simplified fractional Fourier transform (DSFRFT). The proposed definitions can effectively deal with special complex tenors, which further motivates the transform based tensor analysis approaches. Then, we define a new tensor nuclear norm induced by the DSFRFT based t-SVD. In addition, we analyze the computational complexity of the proposed t-SVD, which indicates that the proposed t-SVD can improve the computational efficiency.

Keywords: tensor-tensor product; tensor singular value decomposition; fractional Fourier transform

1 Introduction

With the advance in information science, the data are usually stored in multiway arrays called tensors, which can express more rich and comprehensive information. Tensor analysis is of great importance in various fields, including computation vision [1] and signal processing [2], etc. Since tensors are the generalization of matrices, it seems natural to transform tensors to matrices and apply the existing classical methods to solve the tensor problems. However, these methods can not take advantages of complex structure of tensors and cause the performance degradation. Several tensor factorization (TF) methods and definitions of tensor ranks have been proposed, but each of them has its shortcoming. The Candecomp/Parafac (CP) rank is defined as the minimum number of rank-one tensor decomposition. It is usually NP-hard to calculate and its convex envelop is unclear [3,4]. The Tucker rank is a vector, whose element is matrix rank function of mode-i matricization of a tensor [5]. The sum of the nuclear norm (SNN), as a convex relaxation of the Tucker rank, is widely applied. However, SNN is not the convex envelop of the Tucker rank, which achieves the suboptimal solution substantially [6]. Recently, based on tensor-tensor product (t-product) and tensor singular value decomposition (t-SVD), the tensor nuclear norm (TNN), as the convex envelop of the tensor average rank, is defined in [7,8]. TNN-based models are effective applied in tensor completion [9,10] and tensor robust principal component analysis (PCA) [8]. Note that, based on the discrete Fourier transform (DFT), t-product can be regarded as the convolution-like operation. In addition, for a real tensor the conjugate symmetry of DFT leads to a more efficient method for computing t-product and t-SVD. In [11], the authors proposed a more general t-product and t-SVD based on different real linear transforms. Besides, similar to TNN, the transform

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based TNN is proposed, which is significant in dealing with different real tensors.

In practice, we also need to deal with various complex tensors [12,13]. For example, for chirp complex tensors, chirp parameter estimation based on TF methods have been studied [14]. In order to further improve the efficiency for computing t-product and t-SVD, we propose t-product and t-SVD based on the discrete simplified fractional Fourier transform (DSFRFT). Then, a new tensor nuclear norm is defined based on the proposed t-SVD. These definitions further motivates the transform based tensor analysis approaches. In addition, as the concrete analysis in Section 3, the computation of DSFRFT based t-product and t-SVD are more efficient than those based on DFT. The above definitions based on DSFRFT can reduce to the existing definitions based on DFT when DSFRFT reduces to DFT with the appropriate parameter.

2 TF Induced by DSFRFT

2.1 DSFRFT

The simplified fractional Fourier transform (SFRFT) of $x(t)$ is defined as follows

$$X_p(u)=\sqrt{\frac{1}{\mathrm{j}2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{j}\cdot u\cdot t}\mathrm{e}^{\frac{\mathrm{j}}{2}\cot\alpha\cdot t^2}x(t)\mathrm{d}t \tag{1}$$

where $\alpha=\dfrac{\pi}{2}p$ and $0<|p|<2$ [15,16]. When $p=\pm1$, the SFRFT approximatively reduces to the ordinary Fourier transform (FT). We use $\boldsymbol{S}_n$ to denote the DSFRFT on $\boldsymbol{v}\in\mathbb{C}^n$, and $\widetilde{\boldsymbol{v}}$ is given by $\widetilde{\boldsymbol{v}}=\boldsymbol{S}_n\boldsymbol{v}\in\mathbb{C}^n$. Here, $\boldsymbol{S}_n$ is defined as

$$\boldsymbol{S}_n=\frac{1}{\sqrt{\mathrm{j}}}\boldsymbol{F}_n\boldsymbol{H}_n \tag{2}$$

where $\boldsymbol{F}_n$ is the DFT matrix and $\boldsymbol{H}_n=\mathrm{diag}[1,\mathrm{e}^{\frac{\mathrm{j}}{2}\cot\alpha},\cdots,\mathrm{e}^{\frac{\mathrm{j}}{2}\cot\alpha\cdot(n-1)^2}]$. Based on (2), we present the following conclusions.

Conclusion 1 Let $\boldsymbol{S}_n^*$ be the conjugate transpose of $\boldsymbol{S}_n$, then $\boldsymbol{S}_n^*\boldsymbol{S}_n=\boldsymbol{S}_n\boldsymbol{S}_n^*=n\boldsymbol{I}_n$.

Conclusion 2 For the special complex signal $\boldsymbol{v}=\mathrm{e}^{-\mathrm{j}\gamma k^2}\widehat{\boldsymbol{v}}\in\mathbb{C}^n$, $\widehat{\boldsymbol{v}}\in\mathbb{R}^n$, we obtain $\widetilde{\boldsymbol{v}}=\dfrac{1}{\sqrt{\mathrm{j}}}\overline{\widehat{\boldsymbol{v}}}$, where $\widetilde{\boldsymbol{v}}=\boldsymbol{S}_n\boldsymbol{v}$ and $\overline{\widehat{\boldsymbol{v}}}=\boldsymbol{F}_n\widehat{\boldsymbol{v}}$.

Proof When $\alpha=\mathrm{arccot}\sqrt{2\gamma}$, we have

$$\widetilde{\boldsymbol{v}}=\boldsymbol{S}_n\boldsymbol{v}=\frac{1}{\sqrt{\mathrm{j}}}\boldsymbol{F}_n\boldsymbol{H}_n\boldsymbol{v}=\frac{1}{\sqrt{\mathrm{j}}}\boldsymbol{F}_n\widehat{\boldsymbol{v}}=\frac{1}{\sqrt{\mathrm{j}}}\overline{\widehat{\boldsymbol{v}}} \tag{3}$$

Denote $\mathrm{cicr}(\widehat{\boldsymbol{v}})$ as the circulant matrix of $\widehat{\boldsymbol{v}}$

$$\mathrm{cicr}(\widehat{\boldsymbol{v}})=\begin{bmatrix}\hat{v}_1 & \hat{v}_n & \cdots & \hat{v}_2\\ \hat{v}_2 & \hat{v}_1 & \cdots & \hat{v}_3\\ \vdots & \vdots & \ddots & \vdots\\ \hat{v}_n & \hat{v}_{n-1} & \cdots & \hat{v}_1\end{bmatrix}\in\mathbb{R}^{n\times n} \tag{4}$$

$\mathrm{cicr}(\widehat{\boldsymbol{v}})$ can be diagonalized by $\boldsymbol{F}_n$ as

$$\boldsymbol{F}_n\mathrm{cicr}(\widehat{\boldsymbol{v}})\boldsymbol{F}_n^{-1}=\mathrm{diag}(\overline{\widehat{\boldsymbol{v}}}) \tag{5}$$

where the i-th diagonal element of the diagonal matrix $\mathrm{diag}(\overline{\widehat{\boldsymbol{v}}})$ is $\overline{\widehat{v}}_i$.

Conclusion 3 For the complex signal $\boldsymbol{v}=\mathrm{e}^{-\mathrm{j}\gamma k^2}\widehat{\boldsymbol{v}}\in\mathbb{C}^n$, $\widehat{\boldsymbol{v}}\in\mathbb{R}^n$, we denote $\mathrm{circ}(\boldsymbol{v})$ as the circulant matrix of $\boldsymbol{v}$, and $\mathrm{diag}(\widetilde{\boldsymbol{v}})$ as the diagonal matrix of $\widetilde{\boldsymbol{v}}$. Then, we can deduce the conclusion

$$\boldsymbol{S}_n\cdot(\mathrm{cicr}(\boldsymbol{v})\odot\boldsymbol{G}_n)\cdot\boldsymbol{S}_n^{-1}=\mathrm{diag}(\widetilde{\boldsymbol{v}})$$

where $\boldsymbol{G}_n=\dfrac{1}{\sqrt{\mathrm{j}}}\boldsymbol{H}_n^*\cdot\mathrm{cicr}(\boldsymbol{l}^*)\cdot\boldsymbol{H}_n$ and $\boldsymbol{l}^*=[1,\mathrm{e}^{\mathrm{j}\gamma},\mathrm{e}^{\mathrm{j}4\gamma},\cdots,\mathrm{e}^{\mathrm{j}\gamma(n-1)^2}]^{\mathrm{T}}$.

Proof For the special complex signal $\boldsymbol{v}=\mathrm{e}^{-\mathrm{j}\gamma k^2}\widehat{\boldsymbol{v}}\in\mathbb{C}^n$,$\widehat{\boldsymbol{v}}\in\mathbb{R}^n$, we obtain

$$\mathrm{cicr}(\boldsymbol{v})=\mathrm{cicr}(\boldsymbol{l})\odot\mathrm{cicr}(\widehat{\boldsymbol{v}}) \tag{6}$$

where $\boldsymbol{l}=[1,\mathrm{e}^{-\mathrm{j}\gamma},\mathrm{e}^{-\mathrm{j}4\gamma},\cdots,\mathrm{e}^{-\mathrm{j}\gamma(n-1)^2}]^{\mathrm{T}}$ and $\odot$ is the Hadamard product. As mentioned above, it is known that $\mathrm{circ}(\widehat{\boldsymbol{v}})$ can be diagonalized by $\boldsymbol{F}_n$ as

$$\boldsymbol{F}_n\mathrm{cicr}(\widehat{\boldsymbol{v}})\boldsymbol{F}_n^{-1}=\mathrm{diag}(\overline{\widehat{\boldsymbol{v}}}) \tag{7}$$

Based on (2) and (3), (7) can be convert into

$$\boldsymbol{S}_n\boldsymbol{H}_n^*\mathrm{cicr}(\widehat{\boldsymbol{v}})\boldsymbol{H}_n\boldsymbol{S}_n^{-1}=\mathrm{diag}(\sqrt{\mathrm{j}}\widetilde{\boldsymbol{v}})$$

Use (6), we have

$$\boldsymbol{S}_n\boldsymbol{H}_n^*\mathrm{cicr}(\boldsymbol{v})\odot\mathrm{cicr}(\boldsymbol{l}^*)\boldsymbol{H}_n\boldsymbol{S}_n^{-1}=\mathrm{diag}(\sqrt{\mathrm{j}}\widetilde{\boldsymbol{v}}) \tag{8}$$

Since $\boldsymbol{H}_n$ and $\boldsymbol{H}_n^*$ are diagonal matrixes, based on the property of Hadamard product, (8) can be rewritten as

$$\boldsymbol{S}_n\cdot[\mathrm{cicr}(\boldsymbol{v})\odot\boldsymbol{G}_n]\cdot\boldsymbol{S}_n^{-1}=\mathrm{diag}(\widetilde{\boldsymbol{v}}) \tag{9}$$

where $\boldsymbol{G}_n=\dfrac{1}{\sqrt{\mathrm{j}}}\boldsymbol{H}_n^*\cdot\mathrm{cicr}(\boldsymbol{l}^*)\cdot\boldsymbol{H}_n$.

Specially, when setting $\gamma=\cot\alpha/2$, it is evid-

ent that $\mathrm{diag}(\boldsymbol{l}) = \boldsymbol{H}_n$. When $\alpha = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, $\boldsymbol{H}_n$ is the identity matrix $\boldsymbol{I}_n$, and this property can be regarded as a generalization of that based on DFT.

Conclusion 4 For chirp signal $\boldsymbol{v} = \mathrm{e}^{-\mathrm{j}\gamma k^2}\widehat{\boldsymbol{v}} \in \mathbb{C}^n$, $\widetilde{\boldsymbol{v}} \in \mathbb{R}^n$, the associated $\mathrm{conj}(\widetilde{v}_i) = \sqrt{-1}\widetilde{v}_{n-i+2}$, where $\mathrm{conj}(\widetilde{v}_i)$ is the complex conjugate of $\widetilde{v}_i$.

Proof For $\widehat{\boldsymbol{v}} \in \mathbb{R}^n$, $\overline{\widehat{\boldsymbol{v}}} = \boldsymbol{F}_n\widehat{\boldsymbol{v}}$ satisfies the conjugate symmetry: $\overline{\widehat{v}}_1 \in \mathbb{R}$ and $\mathrm{conj}(\overline{\widehat{v}}_i) = \overline{\widehat{\boldsymbol{v}}}_{n-i+2}$, $i = 2, \ldots, \lfloor\frac{n+1}{2}\rfloor$. Based on Conclusion 2, when $i = 2, \ldots, \lfloor\frac{n+1}{2}\rfloor$, $\mathrm{conj}(\widetilde{v}_i) = \mathrm{conj}(\frac{1}{\sqrt{\mathrm{j}}}) \cdot \widehat{v}_{n-i+2} = \sqrt{-1}\widetilde{v}_{n-i+2}$. Thus, we have the $\mathrm{conj}(\widetilde{v}_1) = \frac{1}{\sqrt{\mathrm{j}}}\widehat{v}_1 \in \mathbb{C}$ and $\mathrm{conj}(\widetilde{v}_i) = \sqrt{-1}\widetilde{v}_{n-i+2}, i = 2, \ldots, \lfloor\frac{n+1}{2}\rfloor$.

2.2 Tensor Nuclear Norm Induced by DSFRFT Based T-SVD

For $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, we denote $\widetilde{\mathcal{A}}$ as the result of DSFRFT applied to each tube $\mathcal{A}(i, j, :)$ and the inverse mapping can be defined in a similar way as follows

$$\begin{aligned} \widetilde{\mathcal{A}} &\doteq \mathcal{S}(\mathcal{A}) = \mathrm{DSFRFT}(\mathcal{A}, [\ ], 3) \\ \mathcal{A} &\doteq \mathcal{S}^{-1}(\widetilde{\mathcal{A}}) = \mathrm{IDSFRFT}(\widetilde{\mathcal{A}}, [\ ], 3) \end{aligned} \quad (10)$$

We define respectively the block diagonal matrix $\mathrm{bdiag}(\widetilde{\mathcal{A}})$ of $\widetilde{\mathcal{A}}$ and the block circulant matrix $\mathrm{bcirc}(\mathcal{A})$ of $\mathcal{A}$ as follows

$$\widetilde{\boldsymbol{A}} = \mathrm{bdiag}(\widetilde{\mathcal{A}}) = \mathrm{diag}[\widetilde{\boldsymbol{A}}^{(1)}, \widetilde{\boldsymbol{A}}^{(2)}, \cdots, \widetilde{\boldsymbol{A}}^{(n_3)}]$$

$$\mathrm{bcirc}(\mathcal{A}) = \begin{bmatrix} \boldsymbol{A}^{(1)} & \boldsymbol{A}^{(n_3)} & \cdots & \boldsymbol{A}^{(2)} \\ \boldsymbol{A}^{(2)} & \boldsymbol{A}^{(1)} & \cdots & \boldsymbol{A}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}^{(n_3)} & \boldsymbol{A}^{(n_3-1)} & \cdots & \boldsymbol{A}^{(1)} \end{bmatrix}$$

By using Conclusion 4, we derive obviously the similar conjugate symmetry of tensors as

$$\begin{cases} \widetilde{\boldsymbol{A}}^{(1)} \in \mathbb{C}^{n_1 \times n_2} \\ \mathrm{conj}(\widetilde{\boldsymbol{A}}^{(i)}) = \sqrt{-1}\widetilde{\boldsymbol{A}}^{(n-i+2)}, i = 2, \ldots, \lfloor\frac{n+1}{2}\rfloor \end{cases} \quad (11)$$

In addition, similar to Conclusion 3, $\mathrm{bcirc}(\mathcal{A})$ can be block diagonalized as

$$(\boldsymbol{S}_{n_3} \otimes \boldsymbol{I}_{n_1}) \cdot (\mathrm{bcirc}(\mathcal{A}) \otimes \boldsymbol{G}_n) \cdot (\boldsymbol{S}_{n_3}^{-1} \otimes \boldsymbol{I}_{n_2}) = \widetilde{\boldsymbol{A}} \quad (12)$$

where $\otimes$ denotes the Kronecker product, $\boldsymbol{I}_{n_1}$ ($\boldsymbol{I}_{n_2}$) denotes the $n_1 \times n_1$ ($n_2 \times n_2$) identity matrix, and $\boldsymbol{G}_n$ is defined in Conclusion 3.

2.3 T-product and T-SVD

For $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, the unfold and fold operator maps are defined respectively as

$$\mathrm{unfold}(\mathcal{A}) = \begin{bmatrix} \boldsymbol{A}^{(1)} \\ \boldsymbol{A}^{(2)} \\ \vdots \\ \boldsymbol{A}^{(n_3)} \end{bmatrix}, \mathrm{fold}(\mathrm{unfold}(\mathcal{A})) = \mathcal{A} \quad (13)$$

Definition 1 (T-product) Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, $\mathcal{B} \in \mathbb{C}^{n_2 \times l \times n_3}$, we denote the t-product $\mathcal{A} *_{\mathcal{S}} \mathcal{B}$ based on DSFRFT as

$$\mathcal{A} *_{\mathcal{S}} \mathcal{B} = \mathrm{fold}((\mathrm{bcirc}(\mathcal{A}) \otimes \boldsymbol{G}_n) \cdot \mathrm{unfold}(\mathcal{B})) \quad (14)$$

When $\mathcal{A}$ is a chirp tensor, $\boldsymbol{G}_n$ is constructed based on the definition in (9). Clearly, when $\alpha = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, $\boldsymbol{H}_n$ is the identity matrix $\boldsymbol{I}_n$. At this moment, DSFRFT reduces to nearly DFT, and the corresponding t-product is DFT-based t-product [8].

Conclusion 5 The DSFRFT-based t-product in Definition 1 satisfy the property, $\mathcal{C} = \mathcal{A} *_{\mathcal{S}} \mathcal{B}$ is equivalent to $\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{B}}$.

Proof

$$\begin{aligned} \mathrm{unfold}(\mathcal{C}) =& \mathrm{unfold}(\mathcal{A} *_{\mathcal{S}} \mathcal{B}) = \\ & (\mathrm{bcirc}(\mathcal{A}) \otimes \boldsymbol{G}_n) \cdot \mathrm{unfold}(\mathcal{B}) = \\ & (\boldsymbol{S}_{n_3}^{-1} \otimes \boldsymbol{I}_{n_1}) \cdot ((\boldsymbol{S}_{n_3} \otimes \boldsymbol{I}_{n_1}) \cdot (\mathrm{bcirc}(\mathcal{A}) \otimes \boldsymbol{G}_n) \cdot \\ & (\boldsymbol{S}_{n_3}^{-1} \otimes \boldsymbol{I}_{n_2})) \cdot (\boldsymbol{S}_{n_3} \otimes \boldsymbol{I}_{n_2}) \cdot \mathrm{unfold}(\mathcal{B}) = \\ & (\boldsymbol{S}_{n_3}^{-1} \otimes \boldsymbol{I}_{n_1}) \cdot \mathrm{bdiag}(\widetilde{\mathcal{A}}) \cdot \mathrm{unfold}(\widetilde{\mathcal{B}}) \end{aligned}$$

Left multiplying both sides with $(\boldsymbol{S}_{n_3^{-1}} \otimes \boldsymbol{I}_{n_1})$ leads to $\mathrm{unfold}(\widetilde{\mathcal{C}}) = \widetilde{\boldsymbol{A}} \odot \mathrm{unfold}(\widetilde{\mathcal{B}})$, which is equivalent to $\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{B}}$.

Definition 2 (Tensor transpose[11]) Let $\mathcal{S}$ be the DSFRFT operator in (10) and $\mathcal{A}^{\intercal}$ be the tensor transpose of $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. Then, $\mathcal{A}^{\intercal}$ satisfies $\mathcal{S}(\mathcal{A}^{\intercal})^{(i)} = (\mathcal{S}(\mathcal{A}^{(i)}))^{\intercal}, i = 1, \cdots, n_3$.

Definition 3 (Identity tensor[11]) Let $\mathcal{I} \in \mathbb{C}^{n \times n \times n_3}$ so that each frontal of $\mathcal{S}(\mathcal{I}) = \overline{\mathcal{I}}$ is a $n \times n$ sized identity matrix. Then, $\mathcal{I} = \mathcal{S}^{-1}(\overline{\mathcal{I}})$ is called the identity tensor.

Definition 4 (Orthogonal tensor[11]) A tensor $\mathcal{Q} \in \mathbb{C}^{n \times n \times n_3}$ is orthogonal if it satisfies $\mathcal{Q}^{\intercal} *_{\mathcal{S}} \mathcal{Q} = \mathcal{Q} *_{\mathcal{S}} \mathcal{Q}^{\intercal}$.

Definition 5 (F-diagonal tensor[11]) A tensor is called f-diagonal if each of its frontal slices is a diagonal matrix.

Theorem 1 (T-SVD) Let $\mathcal{S}$ be the DSFRFT operator in (10) and $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times n_3}$ is a chirp tensor. Then it can be factorized as

$$\mathcal{A}=\mathcal{U}*_{\mathcal{S}}\mathcal{T}*_{\mathcal{S}}\mathcal{V}^{*} \tag{15}$$

where $\mathcal{U}\in\mathbb{C}^{n_1\times n_1\times n_3}$, $\mathcal{V}\in\mathbb{C}^{n_2\times n_2\times n_3}$ and $\mathcal{T}\in\mathbb{R}^{n_1\times n_2\times n_3}$ is an f-diagonal tensor. Based on (11) and (12), a more efficient way to compute DSFRFT based t-SVD is presented in Algorithm 1.

Algorithm 1 T-SVD based on t-product $*_{\mathcal{S}}$

Input: The chirp matrix $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times n_3}$ and the DSFRFT operator $\mathcal{S}$.

Output: T-SVD components $\mathcal{U}$, $\mathcal{T}$ and $\mathcal{V}$ of $\mathcal{A}$.

1) Compute $\widetilde{\mathcal{A}}=\mathcal{S}(\mathcal{A})$.

2) Compute each frontal slice of $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{V}}$ from $\widetilde{\mathcal{A}}$ by

for $i=1,\cdots,\lceil\frac{n_3+1}{2}\rceil$ do

$[\widetilde{\boldsymbol{U}}^{(i)},\widetilde{\boldsymbol{T}}^{(i)},\widetilde{\boldsymbol{V}}^{(i)}]=\mathrm{SVD}(\widetilde{\mathrm{A}}^{(i)})$;

end for

for $i=\lceil\frac{n_3+1}{2}\rceil+1,\cdots,n_3$ do

$\widetilde{\boldsymbol{U}}^{(i)}=\mathrm{conj}(\sqrt{-1}\widetilde{\boldsymbol{U}}^{(n_3-i+2)})$;

$\widetilde{\boldsymbol{T}}^{(i)}=\widetilde{\boldsymbol{T}}^{(n_3-i+2)}$;

$\widetilde{\boldsymbol{V}}^{(i)}=\mathrm{conj}(\sqrt{-1}\widetilde{\boldsymbol{V}}^{(n_3-i+2)})$;

end for

3) Compute $\mathcal{U}=\mathcal{S}^{-1}(\widetilde{\mathcal{U}})$, $\mathcal{T}=\mathcal{S}^{-1}(\widetilde{\mathcal{T}})$, and $\mathcal{V}=\mathcal{S}^{-1}(\widetilde{\mathcal{V}})$.

Definition 6 (Tensor tubal rank[11]) The tensor tubal rank, denoted as $\mathrm{rank}_t(\mathcal{A})$, is defined as the number of nonzero singular tubes of $\mathcal{T}$, where $\mathcal{T}$ is from the t-SVD of $\mathcal{A}=\mathcal{U}*_{\mathcal{S}}\mathcal{T}*_{\mathcal{S}}\mathcal{V}^{*}$, i.e.,

$$\mathrm{rank}_t(\mathcal{A})\doteq\{i,\mathcal{T}(i,i,:)\neq 0\} \tag{16}$$

The tensor tubal rank is nonconvex. As the convex surrogate of $\mathrm{rank}_t(\mathcal{A})$, the convex tensor nuclear norm can be defined inspired by the proof in [11]. As the matrix case, the tensor nuclear norm is the dual norm of the tensor spectral norm. Thus, the definition of the tensor spectral norm is significant.

Definition 7 (Tensor spectral norm) For $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times n_3}$, the tensor spectral norm of $\mathcal{A}$ is defined as $\|\mathcal{A}\|\doteq\|\widetilde{\boldsymbol{A}}\|$.

Since the DSFRFT matrix $\boldsymbol{S}_n$ defined in (2) satisfies $\boldsymbol{S}_n^{*}\boldsymbol{S}_n=\boldsymbol{S}_n\boldsymbol{S}_n^{*}=n\boldsymbol{I}_n$, the corresponding tensor mapping $(\boldsymbol{S}_{n_3}\otimes\boldsymbol{I}_{n_1})/\sqrt{n_3}$ is unitary. Clearly, we have the following property

$$\|\mathcal{A}\|_F=\frac{1}{\sqrt{n_3}}\|\widetilde{\boldsymbol{A}}\|_F,\quad \langle\mathcal{A},\mathcal{B}\rangle=\frac{1}{n_3}\langle\widetilde{\boldsymbol{A}},\widetilde{\boldsymbol{B}}\rangle \tag{17}$$

As the analysis in [8], we present the corresponding analysis based on the DSFRFT. For any $\mathcal{B}\in\mathbb{C}^{n_1\times n_2\times n_3}$ and $\check{\boldsymbol{B}}\in\mathbb{C}^{n_1n_3\times n_2n_3}$, we obtain

$$\begin{aligned}\|\mathcal{A}\|_{*}=&\sup_{\|\mathcal{B}\|\leq 1}\langle\mathcal{A},\mathcal{B}\rangle=\sup_{\|\widetilde{\boldsymbol{B}}\|\leq 1}\frac{1}{n_3}\langle\widetilde{\boldsymbol{A}},\widetilde{\boldsymbol{B}}\rangle\leq\\&\frac{1}{n_3}\sup_{\|\check{\boldsymbol{B}}\|\leq 1}|\langle\widetilde{\boldsymbol{A}},\check{\boldsymbol{B}}\rangle|=\frac{1}{n_3}\|\widetilde{\boldsymbol{A}}\|_{*}\end{aligned} \tag{18}$$

where $\|\widetilde{\boldsymbol{A}}\|_{*}$ is the nuclear norm of matrix $\widetilde{\boldsymbol{A}}$. Let $\mathcal{A}=\mathcal{U}*_{\mathcal{S}}\mathcal{T}*_{\mathcal{S}}\mathcal{V}^{*}$ be the DSFRFT-based t-SVD of $\mathcal{A}$ and $\mathcal{B}=\mathcal{U}*_{\mathcal{S}}\mathcal{V}^{*}$, then we obtain

$$\begin{aligned}\|\mathcal{A}\|_{*}=&\langle\mathcal{U}*_{\mathcal{S}}\mathcal{T}*_{\mathcal{S}}\mathcal{V}^{*},\mathcal{U}*_{\mathcal{S}}\mathcal{V}^{*}\rangle=\\&\frac{1}{n_3}\langle\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{T}}\widetilde{\boldsymbol{V}}^{*},\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{V}}^{*}\rangle=\\&\frac{1}{n_3}\mathrm{Tr}(\widetilde{\boldsymbol{T}})=\frac{1}{n_3}\|\widetilde{\boldsymbol{A}}\|_{*}\end{aligned} \tag{19}$$

Definition 8 (Tensor nuclear norm) For $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times n_3}$, the tensor spectral norm of $\mathcal{A}$ is defined as $\|\mathcal{A}\|_{*}\doteq\frac{1}{n_3}\|\widetilde{\boldsymbol{A}}\|_{*}$.

On the set of $\{\mathcal{A}|\|\mathcal{A}\|\leq 1\}$, the above tensor nuclear norm is the convex envelope of the tensor average rank, defined as $\mathrm{rank}_a(\mathcal{A})=\frac{1}{n_3}\mathrm{rank}(\widetilde{\boldsymbol{A}})$.

3 Computational Complexity

In this section, we compare the computation complexity of t-SVD based on DSFRFT in Algorithm 1 and t-SVD based on DFT for a chirp tensor $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times n_3}$. The proposed t-SVD includes three parts: the DSFRFT, SVD of the transformed matrices and the inverse DSFRFT. We denote $\overline{n}_3=\frac{n_3}{2}\approx\lceil\frac{n_3+1}{2}\rceil$, the multiplications and additions of t-SVD for calculating the

DSFRFT of $\mathcal{A}$ are $2n_1n_2\overline{n}_3 + \frac{1}{2}n_1n_2\overline{n}_3\log_2(n_1n_2)$ and $n_1n_2\overline{n}_3\log_2(n_1n_2)$, respectively. The multiplications and additions of t-SVD for calculating the DFT of $\mathcal{A}$ are $\frac{1}{2}n_1n_2n_3\log_2(n_1n_2)$ and $n_1n_2n_3\log_2(n_1n_2)$, respectively. Obviously, the additions for calculating the DSFRFT of $\mathcal{A}$ is less than those of DFT. When $\log_2(n_1n_2) > 4$, the multiplications for calculating the DSFRFT of $\mathcal{A}$ is also less than those of DFT and the difference is $\log_2\left(\frac{n_1n_2}{16}\right)^{\frac{1}{4}}$. The computation complexity for calculating the SVD of the transformed matrixes and the inverse transform have the similar conclusions. Thus, additions and multiplications of our proposed t-SVD have lower computational cost compared with those of t-SVD based on DFT.

4 Conclusion

In this paper, we present definitions of t-product and t-SVD based on the DSFRFT. By applying the DSFRFT to deal with chirp tensors, the computation of t-product and t-SVD enjoy the conjugate symmetry and can be calculated more efficiently. Then, a new tensor nuclear norm can be defined induced by the DSFRFT based t-SVD. In addition, we analyze the computational complexity of the proposed t-SVD, and present the conclusion that the proposed t-SVD is at lower multiplication and addition complexity than those of the typical DFT based t-SVD.

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