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*Research article*

# Convexity of the free boundary for two-dimensional Réthy flows

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**Abstract:** This paper investigates the convexity of the free boundaries for two-dimensional symmetric Réthy flows. Building on the known existence theory, we consider a steady, incompressible, irrotational flow past an obstacle. Under the assumptions that the upper nozzle wall is horizontal and that the obstacle profile satisfies suitable convexity conditions, we establish a relation between streamline curvature and the normal derivative of the flow speed. Using the subharmonicity of the squared speed and the maximum principle, we prove that the speed in the fluid interior is strictly less than its constant value on the free boundaries. Finally, Hopf's boundary-point lemma and the curvature–speed relation imply that the upper free-boundary graph is strictly convex and that the lower free-boundary graph is strictly concave.

**Keywords:** Réthy flow; free boundary; convexity; maximum principle; Hopf lemma
**Mathematics Subject Classification:** 35Q31, 35J25, 76B10, 35Q35

## 1. Introduction and main results

### *1.1. Introduction*

In 1868, Helmholtz [1] studied flows with free streamlines. Hodograph transformations and conformal mapping theory [2, 3] subsequently became fundamental tools for studying two-dimensional, steady, irrotational jet and cavity flows. Alt and Caffarelli [4] established existence and regularity results for a variational free-boundary problem. Alt, Caffarelli, and Friedman subsequently developed variational approaches to jet and cavity flows [5–8]. Li [9] investigated two-dimensional, compressible, subsonic jet flows with general vorticity governed by the full Euler system. That work established a variational formulation for the stream function and proved the existence and uniqueness of smooth, subsonic jet-flow solutions using the variational method of Alt, Caffarelli, and Friedman. It also characterized the far-field behavior of the flow and established the existence of a critical upstream pressure. Building on the well-posedness results established by Li [9], Wang [10] proved that when the nozzle is concave toward the fluid, the free boundary is strictly

## 4. Conclusions

This paper establishes the convexity of the free boundaries for two-dimensional symmetric Réthy flows with a horizontal upper nozzle wall and an obstacle satisfying the stated geometric assumptions. The proof combines a strict interior speed estimate, a curvature–speed identity, and Hopf's boundary point lemma. The upper free-boundary profile is strictly convex, whereas the lower profile is strictly concave. The analysis is restricted to steady, incompressible, and irrotational flows under the stated geometric assumptions. Further work could investigate analogous geometric properties of free boundaries in flows with nonhorizontal nozzle walls or nonzero vorticity.

## Use of generative-AI tools declaration

The author used Doubao for English translation and language editing during the preparation of this manuscript. The tool was used only to improve grammar, clarity, and readability in the abstract and introduction. All mathematical arguments, results, and proofs were independently developed and verified by the author. The author reviewed and revised all AI-assisted text and takes full responsibility for the content of the manuscript.

## Conflict of interest

The author declares no conflict of interest.

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