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Generalized Uncertainty Inequalities on Fisher Information Associated with LCT

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Abstract: Uncertainty principle plays an important role in multiple fields such as physics, mathematics, signal processing, etc. The linear canonical transform (LCT) has been used widely in optics and information processing and so on. In this paper, a few novel uncertainty inequalities on Fisher information associated with linear canonical transform are deduced. These newly deduced uncertainty relations not only introduce new physical interpretation in signal processing, but also build the relations between the uncertainty lower bounds and the LCT transform parameters a, b, c and d for the first time, which give us the new ideas for the analysis and potential applications. In addition, these new uncertainty inequalities have sharper and tighter bounds which are the generalized versions of the traditional counterparts. Furthermore, some numeric examples are given to demonstrate the efficiency of these newly deduced uncertainty inequalities.

Keywords: linear canonical transform (LCT); Fisher information; uncertainty principle

1 Introduction

Uncertainty principle is a key principle in physics, mathematics, signal processing, etc. [1-23]. Especially, in conventional time-frequency domains, the product of time spread and frequency spread can not be small unlimitedly. On the other hand, if the resolution in one domain increases, then the resolution in another domain necessarily decreases, and vice verse. At the same time, in mathematics, physics and information theory, the Fisher information is the current manner of measuring the amount of information. Based on the Fisher information, some new derived uncertainty relations including the Cramér-Rao inequality have been discussed recently (such as [9-14]). Up till now, in the traditional time-frequency domain, the uncertainty relations have been addressed fully [3-8].

As the generalization of the traditional Fourier transform (FT), Fresnel transform and the fractional Fourier transform (FrFT), the linear canonical transform (LCT) has more freedoms and has been applied in various fields such as physical optics and information science and so on [24-32], which will result in new properties and more applications in signal processing and so on. Therefore, LCT will have more advantages over the traditional counterparts in multiple fields. For more details, see [24,26,27] . On the other hand, this paper is also an improvement of [33,34] for more parameters with new physical interpretation.

We must note that there have been lots of related papers involving generalized uncertainty principles. For example, the entropy based uncertainty principles have been addressed in [12,35] to show the relations between the uncertainty bounds and LCT parameters. The generalized Heisenberg uncertainty principles have been addressed in [15-25] in LCT domains and so on, which have been demonstrated the new physical

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senses on signal resolution analysis. Also, there have been some papers involving the uncertainty relations on Fisher information in traditional time-frequency domain and FRFT domain. However, there have been no any report addressing the uncertainty relations on Fisher information in LCT domain, which will be shown in this paper.

This paper is organized as follows. Section 2 reviews the traditional uncertainty relations, the LCT along with some properties. Then the new generalized Cramér-Rao inequality is proven and explained in the new physical sense as well. In addition, the newly proposed uncertainty relations on Fisher information are discussed. In Section 3, some experiments are given to show the efficiency of the newly derived inequality relations. In Section 4, the results and discussion are given. Finally, our last section will concludes this paper.

2 Proposed Method

2.1 The LCT

For a given function $f(t) \in L^1(R) \cap L^2(R)$ (we assume $\|f(t)\|_2 = 1$ and $f(t)$ is absolutely integrable in this paper) and its traditional Fourier transform $F(u)$, then we have

$$\Delta t^2 \Delta u_{cl}^2 = \int_{-\infty}^{+\infty} |(t-t_0) f(t)|^2 \mathrm{d}t \cdot \int_{-\infty}^{+\infty} |(u-u_0) F(u)|^2 \mathrm{d}u \geqslant \frac{1}{4} \tag{1}$$

where $t_0 = \int_{-\infty}^{+\infty} t|f(t)|^2 \mathrm{d}t$, $u_0 = \int_{-\infty}^{+\infty} u|F(u)|^2 \mathrm{d}u$, Δt^2 is the time spread, and Δu_{cl}^2 is the frequency spread. Set $t_0 = 0$ (for given $f(t)$ we assume $t_0 \equiv 0$ without loss of generality) and $u_0 = 0$, and the essence of uncertainty principle will not change according to [10-23]. Eq. (1) can be rewritten as

$$\Delta t^2 \cdot \Delta u_{cl}^2 = \int_{-\infty}^{+\infty} |tf(t)|^2 \mathrm{d}t \cdot \int_{-\infty}^{+\infty} |uF(u)|^2 \mathrm{d}u \geqslant 1/4 \tag{2}$$

At the same time, for the given real function $f(t)$ defined in above, we have the traditional Cramér–Rao inequality as follows [11-13]

$$\int_{-\infty}^{+\infty} t^2 |f(t)|^2 \mathrm{d}t \cdot \int_{-\infty}^{+\infty} |f(t)|^2 \left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2 \mathrm{d}t \geqslant 1 \tag{3}$$

Furthermore, based on the Fisher information, we also have the uncertainty relation [14]

$$\int_{-\infty}^{+\infty} |F(u)|^2 \left[\frac{\mathrm{d}\ln\left(|F(u)|^2\right)}{du}\right]^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |f(t)|^2 \left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2 \mathrm{d}t \geqslant 4 \tag{4}$$

On the other hand, for the given real function $f(t) \in L^1(R) \cap L^2(R)$ and $\|f(t)\|_2 = 1$, its linear canonical transform (LCT) is defined as [24,26,27]

$$F_A(u) = F_{(a,b,c,d)}(f(t)) = \int_{-\infty}^{\infty} f(t) K_A(u,t) \mathrm{d}t = \begin{cases} \sqrt{\dfrac{1}{\mathrm{i}2\pi b}} \cdot \mathrm{e}^{\frac{\mathrm{i}du^2}{2b}} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-\mathrm{i}ut}{b}} \mathrm{e}^{\frac{\mathrm{i}at^2}{2b}} f(t) \mathrm{d}t \\ b \neq 0, ad - bc = 1 \\ \sqrt{d} \cdot \mathrm{e}^{\frac{\mathrm{i}cdu^2}{2}} f(du), \quad b = 0 \end{cases} \tag{5}$$

where i is the complex unit, $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a, b, c, d \in \mathbf{R}$.

The properties of LCT are shown as follows:

1) Additive property:

$F_{(a_2,b_2,c_2,d_2)}\{F_{(a_1,b_1,c_1,d_1)}(f(t))\} = F_{(e,f,g,h)}(f(t))$ with

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \tag{6}$$

2) Reversibility:

$F_{(a_2,b_2,c_2,d_2)}\{F_{(a_1,b_1,c_1,d_1)}(f(t))\} = f(t)$ with

$$\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

3) Multiplication property:

$$F_{(a,b,c,d)}\{tf(t)\} = duF_{(a,b,c,d)}(u) + \mathrm{i}b\frac{\mathrm{d}F_{(a,b,c,d)}(u)}{du} \tag{8}$$

4) Difference property:

$$\int_{-\infty}^{+\infty} \left|F'_{(a,b,c,d)}(u)\right|^2 \mathrm{d}u = \int_{-\infty}^{+\infty} |\mathrm{d}f'(t) + \mathrm{i}ctf(t)|^2 \mathrm{d}t \tag{9}$$

5) Generalized parseval principle:

$$\int_{-\infty}^{\infty} f(t)g^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} F_{(a,b,c,d)}(u)G^*_{(a,b,c,d)}(u)\mathrm{d}u \tag{10}$$

6) Scaling property:

$$F_{(a,b,c,d)}\left\{\sqrt{\frac{1}{\sigma}}f\left(\frac{t}{\sigma}\right)\right\} = F_{(a\sigma,b/\sigma,c\sigma,d/\sigma)}(u) \tag{11}$$

7) Shift property:

$$F_{(a,b,c,d)}\left\{f(t-\tau)\right\} = \mathrm{e}^{-\mathrm{i}\frac{ac}{2}\tau^2}\mathrm{e}^{\mathrm{i}cu\tau}F_{(a,b,c,d)}(u-a\tau) \tag{12}$$

According to the definition and the properties shown in above, we can obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty} tf(t)f'(t)\mathrm{d}t = -1/2\\ &\int_{-\infty}^{+\infty} f(t)f'(t)\mathrm{d}t = 0\\ &\int_{-\infty}^{+\infty} v|F_A(v)|^2\mathrm{d}v \equiv 0\end{aligned} \tag{13}$$

More details of the properties can be found in [24,26,27].

2.2 The Generalized Cramér-Rao Inequalities on LCT

We know that the traditional Cramér-Rao inequality only involves one single transform domain (either the time domain or the frequency domain) [9-14]. Differently, in this section we will deduce the generalized Cramér-Rao inequality involving both the time domain and the frequency domain (including the generalized frequency domain such as the LCT transform domain) at the same time.

Theorem 1 For the given real function $f(t)$ satisfying $\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t = 1$ and $f(t)$ being absolutely integrable, $F_A(u)$ and $F_B(u)$ are the LCTs of $f(t)$ respectively with the transform parameters $\boldsymbol{A} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $k \in R^+$, then we have

$$\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t \cdot \int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t \geqslant \frac{|a_1b_1a_2c_2|}{2}c_1^2d_1^2b_2^2d_2^2, \text{“=” holds if } f(t) = e^{-kt^2} \text{ and } |a_1b_1a_2c_2|\,c_1^2d_1^2b_2^2d_2^2 = 2. \tag{14}$$

Proof From the traditional Cramér-Rao inequality, we have

$$\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t \cdot \int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t \geqslant 1$$

According to the differential property, we obtain the following relation

$$\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t = \int_{-\infty}^{+\infty}\frac{\left[\left(|f(t)|^2\right)'\right]^2}{|f(t)|^2}\mathrm{d}t = 4\int_{-\infty}^{+\infty}[f'(t)]^2\mathrm{d}t$$

At the same time, we can have

$$\begin{aligned}&\int_{-\infty}^{+\infty}[f'(t)]^2\mathrm{d}t = \int_{-\infty}^{+\infty}\left[F_A\left(f'(t)\right)\right]^2\mathrm{d}t =\\ &\int_{-\infty}^{+\infty}d_1^2\left|F_A'(u)\right|^2 + u^2c_1^2|F_A(u)|^2\mathrm{d}u +\\ &\mathrm{i}c_1d_1\int_{-\infty}^{+\infty}u\left(F_A(u)\left(F_A'(u)\right)^* - F_A'(u)F_A^*(u)\right)\mathrm{d}u =\\ &d_1^2\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u + c_1^2\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u +\\ &\mathrm{i}c_1d_1\int_{-\infty}^{+\infty}u\left(F_A(u)\left(F_A'(u)\right)^* - F_A'(u)F_A^*(u)\right)\mathrm{d}u\end{aligned}$$

Considering the fact that $f(t) \in \mathbf{R}$, then $f'(t) \in \mathbf{R}$, therefore $\int_{-\infty}^{+\infty}[f'(t)]^2\mathrm{d}t \in \mathbf{R}$. At the same time, we also know

$$d_1^2\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u + c_1^2\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u \in \mathbf{R}$$

therefore for any given transform A, we obtain the following relation

$$\mathrm{i}c_1d_1\int_{-\infty}^{+\infty}u\left(F_A(u)\left(F_A'(u)\right)^* - F_A'(u)F_A^*(u)\right)\mathrm{d}u \in \mathbf{R}$$

Therefore, we necessarily have

$$c_1d_1\int_{-\infty}^{+\infty}u\left(F_A(u)\left(F_A'(u)\right)^* - F_A'(u)F_A^*(u)\right)\mathrm{d}u \equiv 0$$

Therefore we have

$$\int_{-\infty}^{+\infty}t^2|f(t)|^2\mathrm{d}t \cdot \left(d_1^2\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u + c_1^2\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right) \geqslant 1$$

$$\text{i.e., } d_1^2\left(\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}t\right)+$$
$$c_1^2\left(\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}t\right)\geqslant 1$$

On the other hand, we have

$$\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t=\int_{-\infty}^{+\infty}|F_B\{tf(t)\}|^2\mathrm{d}t=$$
$$\int_{-\infty}^{+\infty}\left(ud_2\cdot F_B(u)+\mathrm{i}b_2F_B'(u)\right)\cdot$$
$$\left(ud_2\cdot F_B(u)+\mathrm{i}b_2F_B'(u)\right)^*\mathrm{d}u=$$
$$d_2^2\int_{-\infty}^{+\infty}u^2|F_B(u)|^2\mathrm{d}u+b_2^2\int_{-\infty}^{+\infty}\left|F_B'(u)\right|^2\mathrm{d}u+$$
$$\mathrm{i}b_2d_2\int_{-\infty}^{+\infty}u\left(F_B'(u)F_B^*(u)-F_B(u)\left(F_B'(u)\right)^*\right)\mathrm{d}u\quad(15)$$

Considering the fact: $\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t\in\mathbf{R}$, $d_2^2\int_{-\infty}^{+\infty}u^2|F_B(u)|^2du\in\mathbf{R}, b_2^2\int_{-\infty}^{+\infty}\left|F_B'(u)\right|^2du\in\mathbf{R},\beta\in\mathbf{R}$, therefore, we necessarily have $\int_{-\infty}^{+\infty}u\left(F_B'(u)F_B^*(u)-F_B(u)\left(F_B'(u)\right)^*\right)\mathrm{d}u\equiv 0$.

Therefore

$$\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant$$
$$\left(|b_2d_2|\sqrt{\int_{-\infty}^{+\infty}u^2|F_B(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}|F_B'(u)|^2\mathrm{d}u}\right)\cdot$$
$$\left(|c_1d_1|\sqrt{\int_{-\infty}^{+\infty}|F_A'(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u}\right)$$

Here we consider the three uncertainty relations involved in [16], we have

$$\int_{-\infty}^{+\infty}u^2|F_B(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}\left|F_B'(u)\right|^2\mathrm{d}u\geqslant$$
$$\frac{\left(a_2d_2-b_2c_2\right)^2}{4}+\left(a_2c_2\Delta t^2+\frac{b_2d_2}{4\Delta t^2}\right)^2$$

Now if $a_2c_2b_2d_2\geqslant 0$, then we obtain

$$\frac{\left(a_2d_2-b_2c_2\right)^2}{4}+\left(a_2c_2\Delta t^2+\frac{b_2d_2}{4\Delta t^2}\right)^2\geqslant$$
$$\frac{\left(a_2d_2-b_2c_2\right)^2}{4}+a_2c_2b_2d_2=\frac{a_2^2d_2^2+b_2^2c_2^2}{4}$$

On the other hand, if $a_2c_2b_2d_2\leqslant 0$, then we can obtain the follows $\frac{\left(a_2d_2-b_2c_2\right)^2}{4}+\left(a_2c_2\Delta t^2+\frac{b_2d_2}{4\Delta t^2}\right)^2\geqslant\frac{\left(a_2d_2-b_2c_2\right)^2}{4}$ because $\left(a_2c_2\Delta t^2+\frac{b_2d_2}{4\Delta t^2}\right)^2\geqslant 0$, hence we have $\frac{\left(a_2d_2-b_2c_2\right)^2}{4}=\frac{a_2^2d_2^2+b_2^2c_2^2-2a_2c_2b_2d_2}{4}\geqslant\frac{a_2^2d_2^2+b_2^2c_2^2}{4}$ because $-2a_2c_2b_2d_2\geqslant 0$ according to the above hypothesis.

Hence, regardless of the sign of $a_2c_2b_2d_2$, we always have the following relation holding true

$$\sqrt{\int_{-\infty}^{+\infty}u^2|F_B(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}|F_B'(u)|^2\mathrm{d}u}\geqslant$$
$$\frac{\sqrt{a_2^2d_2^2+b_2^2c_2^2}}{2}\geqslant\frac{\sqrt{2}}{2}|a_2b_2c_2d_2|$$

In the same manner, we have the following inequality holding true

$$\sqrt{\int_{-\infty}^{+\infty}|F_A'(u)|^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u}\geqslant$$
$$\frac{\sqrt{a_1^2d_1^2+b_1^2c_1^2}}{2}\geqslant\frac{\sqrt{2}}{2}|a_1b_1c_1d_1|$$

Therefore $\int_{-\infty}^{+\infty} t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant$
$\frac{|a_1b_1c_1d_1|\cdot|a_2b_2c_2d_2|\cdot|c_1d_1b_2d_2|}{2}=\frac{|a_1b_1a_2c_2|}{2}c_1^2d_1^2b_2^2d_2^2.$

The proof is finished.

Obviously, theorem 1 is the generalized version of the traditional Cramér-Rao inequality. In practice, the form of the Cramér-Rao inequality is not very different, but the lower bound becomes very different and sharper or tighter. We easily find that although we do not perform LCT here, which still builds the relation between the Cramér-Rao inequality bound and the LCT transform parameters $\boldsymbol{A}=\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$. In other words, this generalized Cramér-Rao inequality implies the potential advantages in the LCT domain over the traditional domain in the point of the parameter estimation and so on.

Theorem 2 For a given real function $f(t)$ satisfying $\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t=1$ and $f(t)$ being absolutely in-

tegrable, $F_A(u)$ is the LCT of $f(t)$ with the transform parameter $\boldsymbol{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then we have

$$\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2\geqslant \max\left\{\frac{(ad-bc)^2}{2}+2\left(ac\Delta t^2+\frac{bd}{4\Delta t^2}\right)^2,\ \frac{1}{d^4+\delta^4}\right\} \text{ with } \delta=\max\{|b|,|c|\}. \tag{16}$$

Proof On one hand, according to the inequality $a^2+b^2\geqslant 2ab,(a,b\in\mathbf{R})$ and based on the result in [16] and the above result, we can obtain the following inequality

$$\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2\geqslant 2\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\geqslant \frac{(ad-bc)^2}{2}+2\left(ac\Delta t^2+\frac{bd}{4\Delta t^2}\right)^2$$

On the other hand, in (15), we let $\boldsymbol{A}=\boldsymbol{B}$, then we have

$$\int_{-\infty}^{+\infty}t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t=(d^4+b^2c^2)\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u+d^2c^2\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+b^2d^2\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\leqslant \frac{d^4+b^2c^2}{2}\left\{\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\right\}+d^2e^2\left\{\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\right\}=\left(\frac{d^4+b^2c^2}{2}+d^2\delta^2\right)\cdot\left\{\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\right\}$$

with $\delta=\max\{|b|,|c|\}$.

Therefore, we have

$$\left(\frac{d^4+b^2c^2}{2}+d^2\delta^2\right)\left\{\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\right\}\geqslant\int_{-\infty}^{+\infty}t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant 1$$

Therefore

$$\left(\frac{d^4+b^2c^2}{2}+d^2\delta^2\right)\left\{\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\right\}\geqslant 1$$

So we have

$$\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\geqslant\frac{2}{d^4+b^2c^2+2d^2\delta^2}$$

At the same time, we know $2d^2\delta^2\leqslant d^4+\delta^4$ and $b^2c^2\leqslant\delta^4$ because $\delta=\max\{b,c\}$, therefore

$$\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2\geqslant\frac{1}{d^4+\delta^4}$$

Finally

$$\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2\geqslant\max\left\{\frac{(ad-bc)^2}{2}+2\left(ac\Delta t^2+\frac{bd}{4\Delta t^2}\right)^2,\ \frac{1}{d^4+\delta^4}\right\}$$

The proof is finished.

Note that Theorem 2 is the generalized Cramér-Rao uncertainty inequality, which only involves the single transform domain (in one single LCT domain). In other words, Theorem 2 is a new uncertainty relation involving only the single transform domain. For the given function $f(t)$ defined in above, its LCT 's variance and LCT 's gradient integral must satisfy certain equality relation, i.e., their square sum cannot be less than some low bound. The biggest novelty here is that this uncertainty inequality builds the relation between the four parameters (a,b,c,d)

and the lower bound, which is sharper and tighter than the traditional counterparts because the new bound is more refined with more parameters.

2.3 Fisher Information Based Generalized Uncertainty Relation on LCT

Like the Shannon entropy and Renyi entropy, the Fisher information is another significant information measure and plays an important role in signal processing. In the same manner, the uncertainty relation based on the Fisher information has the important effect on information processing and signal processing. In this section, we will deduce the Fisher information based generalized uncertainty inequalities in the LCT domain and explain the corresponding physical significance.

Theorem 3 For the given real function $f(t)$ satisfying $\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t=1$ and $f(t)$ being absolutely integrable, $F_A(u)$, $F_B(u)$ are the LCTs of $f(t)$ with the transform parameters $\boldsymbol{A}=\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$, thus we have

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot \int_{-\infty}^{+\infty}|F_B(v)|^2\left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v\geqslant\frac{4}{|a_1b_2-a_2b_1|^2} \tag{17}$$

Proof Based on the Fisher information, we can obtain the traditional uncertainty inequality as follows

$$\int_{-\infty}^{+\infty}|F(u)|^2\left[\frac{\mathrm{d}\ln\left(|F(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot \int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant 4$$

Now set $G(u)=F_A(u)\exp\left(-\mathrm{i}\dfrac{a_3u^2}{2b_3}\right)$, $F_A(u)=F_A\left(f(t)\right)$, $g(t)=\sqrt{\dfrac{1}{2\pi}}\int_{-\infty}^{\infty}G(u)\mathrm{e}^{iut}\mathrm{d}u$. At the same time, considering the relation $\left|F_A(u)\exp\left(-\mathrm{i}\dfrac{a_3u^2}{2b_3}\right)\right|=|F_A(u)|$, we have

$$\int_{-\infty}^{+\infty}|G(u)|^2\left[\frac{\mathrm{d}\ln\left(|G(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u=\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u$$

Therefore

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot \int_{-\infty}^{+\infty}|g(t)|^2\left[\frac{\mathrm{d}\ln\left(|g(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant 4$$

According to the scaling property of $g(t)$, set $\boldsymbol{C}=\begin{bmatrix}a_3 & b_3\\ c_3 & d_3\end{bmatrix}$, we have $\boldsymbol{C}^{-1}=\begin{bmatrix}d_3 & -b_3\\ -c_3 & a_3\end{bmatrix}$ and

$$\int_{-\infty}^{+\infty}\left|g\left(\frac{t}{b_3}\right)\right|^2\left[\frac{\mathrm{d}\ln\left(\left|g\left(\frac{t}{b_3}\right)\right|^2\right)}{\mathrm{d}\left(\frac{t}{b_3}\right)}\right]^2\mathrm{d}\left(\frac{t}{b_3}\right)=\frac{1}{|b_3|}\int_{-\infty}^{+\infty}\left|g\left(\frac{t}{b_3}\right)\right|^2\left[\frac{\mathrm{d}\ln\left(\left|g\left(\frac{t}{b_3}\right)\right|^2\right)}{\mathrm{d}\left(\frac{t}{b_3}\right)}\right]^2\mathrm{d}t$$

According to the definition of Fourier transform, there exists the relation $\left|g\left(\dfrac{t}{b_3}\right)\right|^2=\left|\sqrt{\dfrac{1}{2\pi}}\int_{-\infty}^{\infty}G(u)\mathrm{e}^{\mathrm{i}\frac{ut}{b_3}}\mathrm{d}u\right|^2$. Let $F_A(u)\exp\left(-\mathrm{i}\dfrac{a_3u^2}{2b_3}\right)=G(u)$, we can obtain

$$\left|g\left(\frac{t}{b_3}\right)\right|^2=|b_3|\cdot|F_{C^{-1}}\left\{F_A(u)\right\}(t)|^2$$

Therefore, we obtain

$$\frac{1}{|b_3|}\int_{-\infty}^{+\infty}\left|g\left(\frac{t}{b_3}\right)\right|^2\left[\frac{\mathrm{d}\ln\left(\left|g\left(\frac{t}{b_3}\right)\right|^2\right)}{\mathrm{d}\left(\frac{t}{b_3}\right)}\right]^2\mathrm{d}t=|b_3|^2\int_{-\infty}^{+\infty}|F_{C^{-1}}\left\{F_A(u)\right\}(t)|^2\left[\frac{\mathrm{d}\ln\left(|b_3|\cdot|F_{C^{-1}}\left\{F_A(u)\right\}(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t=|b_3|^2\int_{-\infty}^{+\infty}|F_{C^{-1}A}(v)|^2\left[\frac{\mathrm{d}\ln\left(|b_3|\cdot|F_{C^{-1}A}(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v$$

Here we know that

$$\ln\left(|b_3|\cdot|F_{C^{-1}A}(v)|^2\right)=\ln(|b_3|)+\ln\left(|F_{C^{-1}A}(v)|^2\right)$$
$$\frac{\mathrm{d}\left[\ln(|b_3|)+\ln\left(|F_{C^{-1}A}(v)|^2\right)\right]}{\mathrm{d}v}=\frac{\mathrm{d}\ln\left(|F_{C^{-1}A}(v)|^2\right)}{\mathrm{d}v}$$

Therefore we obtain

$$\frac{1}{|b_3|}\int_{-\infty}^{+\infty}\left|g\left(\frac{t}{b_3}\right)\right|^2\left[\frac{\mathrm{d}\ln\left(\left|g\left(\frac{t}{b_3}\right)\right|^2\right)}{\mathrm{d}\left(\frac{t}{b_3}\right)}\right]^2\mathrm{d}t=$$
$$|b_3|^2\int_{-\infty}^{+\infty}|F_{C^{-1}A}(v)|^2\left[\frac{\mathrm{d}\ln\left(|F_{C^{-1}A}(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v$$

Here we perform the operations: setting $t=v$ and applying $F_{C^{-1}}\{F_A(u)\}(t)=F_{AC^{-1}}(t)$, at the same time letting $\boldsymbol{B}=\boldsymbol{AC}^{-1}$, so we have $\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}=\begin{bmatrix}d_3 & -b_3\\ -c_3 & a_3\end{bmatrix}\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$, then

$$\begin{bmatrix}d_3 & -b_3\\ -c_3 & a_3\end{bmatrix}=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}\begin{bmatrix}d_1 & -b_1\\ -c_1 & a_1\end{bmatrix}=$$
$$\begin{bmatrix}d_1a_2-c_1b_2 & a_1b_2-a_2b_1\\ c_2d_1-c_1d_2 & a_1d_2-b_1c_2\end{bmatrix}$$

Therefore we have $-b_3=a_1b_2-a_2b_1$.

Based on the above equations we can obtain

$$\int_{-\infty}^{+\infty}|g(t)|^2\left[\frac{\mathrm{d}\ln\left(|g(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t=$$
$$|a_1b_2-a_2b_1|^2\int_{-\infty}^{+\infty}|F_B(v)|^2\left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v$$

Finally, we obtain

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot$$
$$\int_{-\infty}^{+\infty}|F_B(v)|^2\left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v\geqslant\frac{4}{|a_1b_2-a_2b_1|^2}$$

The proof is finished.

Clearly, compared with the traditional uncertainty relation on Fisher information, Theorem 3 yields a sharper and tighter bound for the generalized uncertainty relation on Fisher information in the LCT domain with the transform parameters $\boldsymbol{A}=\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$.

Furthermore, if $|a_1b_2-a_2b_1|$, that is to say, the two LCTs are orthogonal for their transform parameters, then $\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}|F_B(v)|^2\left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v\geqslant 4$.

If $(a_1,b_1,c_1,d_1)=(1,0,0,1)$ and $(a_2,b_2,c_2,d_2)=(0,1,-1,0)$, this uncertainty inequality reduces to the traditional case.

Especially, if $(a_2,b_2,c_2,d_2)=(1,0,0,1)$, then we can obtain the following lemma.

Lemma 1 For the given real function $f(t)$ satisfying $\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t=1$ and $f(t)$ being absolutely integrable, $F_A(u)$ is the LCT of $f(t)$ with the transform parameter $\boldsymbol{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then we have

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}[tf(t)]^2\mathrm{d}t\geqslant\frac{1}{b^2}\tag{18}$$

Proof If $(a_2,b_2,c_2,d_2)=(1,0,0,1)$, based on (17), we have

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot$$
$$\int_{-\infty}^{+\infty}|F(v)|^2\left[\frac{\mathrm{d}\ln\left(|F(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v\geqslant\frac{4}{b^2}$$

On the other hand, based on the definitions of LCT and the properties shown in above, we can have

$$\int_{-\infty}^{+\infty}|F(v)|^2\left[\frac{\mathrm{d}\ln\left(|F(v)|^2\right)}{dv}\right]^2\mathrm{d}v=\int_{-\infty}^{+\infty}\frac{\left[\left(F|(v)|^2\right)'\right]^2}{|F(v)|^2}\mathrm{d}v=$$
$$4\int_{-\infty}^{+\infty}[F'(v)]^2\mathrm{d}v,\ \int_{-\infty}^{+\infty}[F'(v)]^2\mathrm{d}v=\int_{-\infty}^{+\infty}[tf(t)]^2\mathrm{d}t$$

Therefore

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot$$
$$\int_{-\infty}^{+\infty}|F(v)|^2\left[\frac{\mathrm{d}\ln\left(|F(v)|^2\right)}{\mathrm{d}v}\right]^2\mathrm{d}v=$$
$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{du}\right]^2\mathrm{d}u\cdot$$
$$4\int_{-\infty}^{+\infty}[tf(t)]^2\mathrm{d}t\geqslant\frac{4}{b^2}$$

Finally, we obtain

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u\cdot\int_{-\infty}^{+\infty}[tf(t)]^2\mathrm{d}t\geqslant\frac{1}{b^2}$$

Up till now, the proof is finished.

Lemma 1 is another new version of the generalized uncertainty relation on the Fisher information. This lemma builds the new relation between the variance, the Fisher information on the LCT of the function $f(t)$ and the LCT transform parameter b for the first time.

Similarly, since there are more parameters in these new bounds than their traditional counterparts, the bounds will be more related with the transform with tighter and sharper values.

3 Experiments

In this section, in order to verify the newly deduced inequalities, we will give some examples. We first let $f(t)=(2/\sqrt{\pi})^{\frac{1}{2}}t\mathrm{e}^{-\frac{t^2}{2}}$, and we have $F_A(u)=\mathrm{e}^{-\mathrm{i}\pi/4}(2/\sqrt{\pi})^{\frac{1}{2}}u\mathrm{e}^{-\frac{u^2}{2}}$ and $F_B(v)=\mathrm{e}^{-\mathrm{i}3\pi/4}\cdot(2/\sqrt{\pi})^{\frac{1}{2}}v\mathrm{e}^{-\frac{v^2}{2}}$. Now we let $\boldsymbol{A}=\frac{\sqrt{2}}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ and $\boldsymbol{B}=\frac{\sqrt{2}}{2}\begin{bmatrix}-1&1\\-1&-1\end{bmatrix}$, then we can obtain

$$\Delta t^2=\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t=\int_{-\infty}^{+\infty}t^2\left|(2/\sqrt{\pi})^{\frac{1}{2}}t\mathrm{e}^{-\frac{t^2}{2}}\right|^2\mathrm{d}t=\frac{3}{2}$$

$$\Delta u_A{}^2=\int_{-\infty}^{+\infty}|uF_A(u)|^2\mathrm{d}u=\int_{-\infty}^{+\infty}u^2\left|\mathrm{e}^{-\mathrm{i}\pi/4}(2/\sqrt{\pi})^{\frac{1}{2}}u\mathrm{e}^{-\frac{u^2}{2}}\right|^2\mathrm{d}u=\frac{3}{2}$$

$$\Delta u_B{}^2=\int_{-\infty}^{+\infty}|vF_B(v)|^2\mathrm{d}v=\int_{-\infty}^{+\infty}v^2\left|\mathrm{e}^{-\mathrm{i}3\pi/4}(2/\sqrt{\pi})^{\frac{1}{2}}v\mathrm{e}^{-\frac{v^2}{2}}\right|^2\mathrm{d}v=\frac{3}{2}$$

$$\int_{-\infty}^{+\infty}\frac{\left[\left(F|(u)|^2\right)'\right]^2}{|F(u)|^2}\mathrm{d}u=6\qquad(19)$$

$$\int_{-\infty}^{+\infty}\frac{\left[\left(f|(t)|^2\right)'\right]^2}{|f(t)|^2}\mathrm{d}t=6\qquad(20)$$

At the same time, considering $\boldsymbol{A}=\frac{\sqrt{2}}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ and $\boldsymbol{B}=\frac{\sqrt{2}}{2}\begin{bmatrix}-1&1\\-1&-1\end{bmatrix}$, we obtain

$$\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u=\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u=\frac{3}{2}\qquad(21)$$

$$\int_{-\infty}^{+\infty}\left|F_B'(v)\right|^2\mathrm{d}v=\int_{-\infty}^{+\infty}v^2|F_B(v)|^2\mathrm{d}v=\frac{3}{2}\qquad(22)$$

Therefore, we have

$$\int_{-\infty}^{+\infty}|F_A(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u=4\int_{-\infty}^{+\infty}\left[F_A'(u)\right]^2\mathrm{d}t=6$$

At the same manner, we can obtain

$$\int_{-\infty}^{+\infty}|F_B(u)|^2\left[\frac{\mathrm{d}\ln\left(|F_B(u)|^2\right)}{\mathrm{d}u}\right]^2\mathrm{d}u=4\int_{-\infty}^{+\infty}\left[F_B'(u)\right]^2\mathrm{d}t=6\qquad(23)$$

For Theorem 1, based on the above equation we have

$$\int_{-\infty}^{+\infty}t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t=\frac{3}{2}\times6=9$$

For Theorem 1, based on the above given parameters for the right side of (14), we have

$$\frac{|a_1b_1a_2c_2|}{2}c_1^2d_1^2b_2^2d_2^2=\frac{1}{8}\times\frac{1}{16}=\frac{1}{128}$$

Therefore, we obtain

$$9=\int_{-\infty}^{+\infty}t^2|f(t)|^2\mathrm{d}t\cdot\int_{-\infty}^{+\infty}|f(t)|^2\left[\frac{\mathrm{d}\ln\left(|f(t)|^2\right)}{\mathrm{d}t}\right]^2\mathrm{d}t\geqslant\frac{|a_1b_1a_2c_2|}{2}c_1^2d_1^2b_2^2d_2^2=\frac{1}{128}$$

which means that Theorem 1 holds true.

In the same manner, for Theorem 2 we can obtain

The left side of (16):

$$\left(\int_{-\infty}^{+\infty}\left|F_A'(u)\right|^2\mathrm{d}u\right)^2+\left(\int_{-\infty}^{+\infty}u^2|F_A(u)|^2\mathrm{d}u\right)^2=\frac{9}{2}$$

The right side of (16):

$$\frac{(a_1d_1-b_1c_1)^2}{2}+2\left(a_1c_1\Delta t^2+\frac{b_1d_1}{4\Delta t^2}\right)^2=\frac{25}{18}$$

$$\frac{1}{d^4+\delta^4}=2$$

Therefore, $\frac{9}{2} \geqslant \max\left\{\frac{25}{18}, 2\right\} = 2$.

Therefore, the inequality in Theorem 2 holds true.

Similarly, for Theorem 3, we have

The left side of (17):

$$\int_{-\infty}^{+\infty} |F_A(u)|^2 \left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |F_B(v)|^2 \left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2 \mathrm{d}v = 36$$

The right side of (17): $\frac{4}{|a_1 b_2 - a_2 b_1|^2} = 4$.

Clearly

$$36 = \int_{-\infty}^{+\infty} |F_A(u)|^2 \left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} |F_B(v)|^2 \left[\frac{\mathrm{d}\ln\left(|F_B(v)|^2\right)}{\mathrm{d}v}\right]^2 \mathrm{d}v \geqslant \frac{4}{|a_1 b_2 - a_2 b_1|^2} = 4$$

which means that Theorem 3 holds true.

For Lemma 1, we have

The left side of (18):

$$\int_{-\infty}^{+\infty} |F_A(u)|^2 \left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} [tf(t)]^2 \mathrm{d}t = 9$$

The right side of (17): $\frac{1}{\cos^2\alpha} = 2$.

Clearly

$$9 = \int_{-\infty}^{+\infty} |F_A(u)|^2 \left[\frac{\mathrm{d}\ln\left(|F_A(u)|^2\right)}{\mathrm{d}u}\right]^2 \mathrm{d}u \cdot \int_{-\infty}^{+\infty} [tf(t)]^2 \mathrm{d}t \geqslant \frac{1}{b_1^2} = 2$$

which means that Lemma 1 holds true.

Furthermore, we also have tried other real function $f(t)$ with different forms, and the same conclusion has been obtained as well. Taking into account of the limited length of this paper, we only show the above examples. In other words, these examples prove that the newly deduced Cramér-Rao inequalities and the uncertainty inequalities based on the Fisher information hold true, which will give us new potential values in information processing and signal processing and so on.

4 Results and Discussion

As shown in Section 3, these experiments prove that the newly deduced Cramér-Rao inequalities and the uncertainty inequalities based on the Fisher information hold true, which will give us new potential values in information processing and signal processing and so on.

On one hand, these novel uncertainty inequalities are the extension of the traditional Cramér-Rao inequality and the uncertainty relation on Fisher information. At the same time, the generalized Cramér-Rao inequalities' bounds are much sharper and tighter than the traditional counterparts.

On the other hand, the generalized Cramér-Rao uncertainty inequalities build the relation between the Cramér-Rao bounds and the LCT transform parameters. In addition, the generalized Cramér-Rao inequalities build the relation between the LCT's variance and LCT's gradient integral in only one single transform domain. Also, compared with the traditional counterparts, the newly deduced uncertainty relations on Fisher information have the sharper and tighter bounds, which show the potential advantage for the parameter estimation in the LCT domain.

5 Conclusion

In this paper, a few novel uncertainty inequalities on Fisher information associated with linear canonical transform are deduced. These newly deduced uncertainty relations make full use of the properties of LCT with more freedoms that will yield more potential values. That is to say, these new uncertainty inequalities employ every parameter of (a, b, c, d). In addition, in our paper some numeric experiments are given to show the efficiency of these newly deduced inequalities. Our future work will include the relations on the complex functions and other more complicated

case such as the Hilbert transform on LCT [34-36] including some equality conditions of these theorems and so on.

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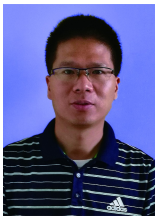

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