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一类分数阶 q-差分方程广义反周期边值问题

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摘要: 考虑一类非线性 Caputo 型分数阶 q-差分方程的广义反周期边值问题, 用 Banach 不动点定理给出该广义反周期边值问题解的存在唯一性结果, 并给出一个应用实例.

关键词: Caputo 分数阶 q-导数; 分数阶 q-差分方程; 广义反周期边值问题; Banach 不动点定理
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Generalized Anti-periodic Boundary Value Problem for a Class of Fractional q-Difference Equations

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Abstract: We considered the generalized anti-periodic boundary value problem for a class of nonlinear Caputo fractional q-difference equations, gave the existence and uniqueness results of solutions for the generalized anti-periodic boundary value problem by using the Banach fixed point theorem, and gave an application example.

Keywords: Caputo fractional q-derivative; fractional q-difference equation; generalized anti-periodic boundary value problem; Banach fixed point theorem

0 引 言

分数阶 q-差分理论[1-2]是分数阶差分体系中的一种特殊形式, 它具有分数阶微积分和离散数学二者的优点, 因而有更丰富的理论意义和应用价值. 目前, 分数阶 q-差分方程的研究主要侧重于 Caputo 分数阶 q-导数和 Riemann-Liouville 分数阶 q-导数两方面. 文献[3]应用锥上不动点定理研究了二阶 q-差分方程边值问题正解的存在性; 文献[4-5]应用 Banach 不动点定理和 Krasnoselskii 不动点定理给出了带有非局部 Riemann-Liouville 分数阶 q-积分边值条件的 Riemann-Liouville 分数阶 q-差分边值问题解的存在性结果; 文献[6-7]应用 Banach 不动点定理和 Covitz-Nadler 不动点定理研究了边值条件含积分的非线性 Caputo 分数阶 q-差分方程以及 q-差分包含边值问解的存在性. 由于反周期问题在许多物理过程的数学模型中应用广泛, 所以反周期边值问题是一类重要的边值问题. 关于非线性分数阶 q-差分方程反周期边值问题的研究已取得了一些进展, 文献[8-10]利用基本的不动点定理研究了一类带有反周期非线性 Caputo 分数阶 q-差分方程边值问题, 得到了边值问题解的存在性和唯一性的

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充分条件；文献[11-12]利用 Banach 不动点定理和 Leary-Schauder 非线性抉择研究了一类带有反周期边值条件的非线性分数阶脉冲 q-差分方程的边值问题，给出了该边值问题解的存在性和唯一性结果.

本文考虑如下非线性项和边值条件都带 Caputo 分数阶 q-导数的非线性分数阶 q-差分方程广义反周期边值问题

$$\begin{cases} ({}^cD_q^\alpha u)(t)=f(t,u(t),({}^cD_q^\beta u)(t)), & 0<t<1,\\ u(0)+au(1)=0, \quad ({}^cD_q^\gamma u)(0)+b({}^cD_q^\gamma u)(1)=\lambda \end{cases} \tag{1}$$

解的存在唯一性，其中 ${}^cD_q^\alpha$ 表示 α 阶 Caputo 分数阶 q-导数，$q\in(0,1)$，$1<\alpha\leqslant 2$，$0<\beta,\gamma\leqslant 1$；$f\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$，$a,b>0$，$\lambda\in\mathbb{R}$. 若 $a=b=1$，$\lambda=0$，则边值问题(1)成为反周期边值问题.

1 预备知识

函数 g 的 q-导数定义为

$$(D_qg)(x)=\frac{g(x)-g(qx)}{(1-q)x}, \qquad (D_qg)(0)=\lim_{x\to 0}(D_qg)(x);$$

g 在[0,1]上的 q-积分定义为

$$(I_qg)(x)=\int_0^x g(t)\mathrm{d}_qt=x(1-q)\sum_{n=0}^{\infty}g(xq^n)q^n, \qquad x\in[0,1].$$

再定义 $(D_q^0g)(x)=g(x)$，$(D_q^ng)(x)=D_q(D_q^{n-1}g)(x)$，$n\in\mathbb{N}$；$(I_q^0g)(x)=g(x)$，$(I_q^ng)(x)=I_q(I_q^{n-1}g)(x)$，$n\in\mathbb{N}$.

设函数 g 定义在[0,1]上，$\alpha\geqslant 0$，则函数 g 的 Riemann-Liouville 分数阶 q-积分定义为

$$(I_q^0g)(x)=g(x), \quad (I_q^\alpha g)(x)=\int_0^x\frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}g(s)\mathrm{d}_qs, \quad \alpha>0, \quad x\in[0,1];$$

函数 g 的 Caputo 分数阶 q-导数定义为

$$({}^cD_q^0g)(x)=g(x), \quad ({}^cD_q^\alpha g)(x)=(I_q^{m-\alpha}D_q^mg)(x), \quad \alpha>0,$$

其中 m 为不小于 α 的最小整数. 文献[13]给出了分数阶 q-积分和 q-导数的基本概念及相关结果.

引理 1 设函数 $h\in C([0,1],\mathbb{R})$，则边值问题

$$\begin{cases} ({}^cD_q^\alpha u)(t)=h(t), & 0<t<1, \quad 1<\alpha\leqslant 2,\\ u(0)+au(1)=0, \quad ({}^cD_q^\gamma u)(0)+b({}^cD_q^\gamma u)(1)=\lambda \end{cases}$$

有唯一解

$$\begin{aligned} u(t)=&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}h(s)\mathrm{d}_qs-\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}h(s)\mathrm{d}_qs-\\ &\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)b}+\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}h(s)\mathrm{d}_qs+\\ &t\left[\frac{\lambda}{b}\Gamma_q(2-\gamma)-\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}h(s)\mathrm{d}_qs\right]. \end{aligned}$$

证明：因为

$$u(t)=\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)\mathrm{d}_qs+c_0+c_1t, \qquad ({}^cD_q^\gamma u)(t)=\int_0^t\frac{(t-qs)^{(\alpha-\gamma-1)}}{\Gamma_q(\alpha-\gamma)}h(s)\mathrm{d}_qs+\frac{c_1t^{(1-\gamma)}}{\Gamma_q(2-\gamma)},$$

故利用边值条件 $u(0)+au(1)=0$，$({}^cD_q^\gamma u)(0)+b({}^cD_q^\gamma u)(1)=\lambda$，可得

$$c_0=-\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}h(s)\mathrm{d}_qs-\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)b}+\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}h(s)\mathrm{d}_qs,$$

$$c_1=\frac{\lambda}{b}\Gamma_q(2-\gamma)-\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}h(s)\mathrm{d}_qs,$$

将其代入 $u(t)$ 中即得结论.

2 主要结果

对于 $U=\{u:\ u\in C([0,1],\mathbb{R}),\ {}^cD_q^\beta u\in C([0,1],\mathbb{R})\}$，规定范数 $\|u\|=\max\limits_{t\in[0,1]}|u(t)|+$

$\max\limits_{t\in[0,1]}|({}^cD_q^\beta u)(t)|$，则 $(U,\|\cdot\|)$ 是一个 Banach 空间. 对任意的 $u\in U$，$t\in[0,1]$，定义算子 $\Phi: U\to U$ 为

$$\begin{aligned}(\Phi u)(t)=&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}f(s,u(s),({}^cD_q^\beta u)(s))\mathrm{d}_qs-\\&\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}f(s,u(s),({}^cD_q^\beta u)(s))\mathrm{d}_qs-\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)b}+\\&\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}f(s,u(s),({}^cD_q^\beta u)(s))\mathrm{d}_qs+\\&t\left[\frac{\lambda}{b}\Gamma_q(2-\gamma)-\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}f(s,u(s),({}^cD_q^\beta)u(s))\mathrm{d}_qs\right],\end{aligned}$$

则

$$\begin{aligned}({}^cD_q^\beta(\Phi u))(t)=&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}f(s,u(s),({}^cD_q^\beta u)(s))\mathrm{d}_qs+\\&\frac{t^{(1-\beta)}}{\Gamma_q(2-\beta)}\left[\frac{\lambda}{b}\Gamma_q(2-\gamma)-\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}f(s,u(s),({}^cD_q^\beta u)(s))\mathrm{d}_qs\right].\end{aligned}$$

由引理 1 知算子 Φ 的不动点即为边值问题(1)的解.

定理 1　设存在常数 $K>0$，使得

$$|f(t,x,y)-f(t,\overline{x},\overline{y})|\leqslant K(|x-\overline{x}|+|y-\overline{y}|),\quad t\in[0,1],\quad x,y,\overline{x},\overline{y}\in\mathbb{R}.\tag{2}$$

若

$$\rho=K\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]<1,$$

则边值问题(1)存在唯一解.

证明：对任意的 $t\in[0,1]$，$u,v\in U$，有

$$\begin{aligned}|(\Phi u)(t)-(\Phi v)(t)|\leqslant&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs+\\&\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs+\\&\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs+\\&\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^t(t-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs\leqslant\\&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}K(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_qs+\\&\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}K(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_qs+\\&\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}K(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_qs+\\&\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}K(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_qs\leqslant\\&K\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}\right]\|u-v\|.\end{aligned}$$

因为

$$\begin{aligned}|({}^cD_q^\beta(\Phi u))(t)-({}^cD_q^\beta(\Phi v))(t)|\leqslant&\frac{1}{\Gamma_q(\alpha-\beta)}\times\\&\int_0^t(t-qs)^{(\alpha-\beta-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs+\\&\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_qs\leqslant\end{aligned}$$

$$K\left[\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\|u-v\|,$$

所以

$$\|\Phi u-\Phi v\|\leqslant K\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\times\|u-v\|=\rho\|u-v\|.$$

又因为 $\rho<1$，所以 Φ 是压缩映射，根据 Banach 不动点定理知，系统(1)存在唯一解.

定理 2 设存在常数 $L,R>0$，使得

$$|f(t,x,y)-f(t,\overline{x},\overline{y})|\leqslant L(|x-\overline{x}|+|y-\overline{y}|),\ t\in[0,1],\ |x|+|y|\leqslant R,\ |\overline{x}|+|\overline{y}|\leqslant R.\quad(3)$$

若

$$\theta=\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\times\left[L+\frac{1}{R}\max_{t\in[0,1]}|f(t,0,0)|\right]+\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)bR}+\frac{\lambda[\Gamma_q(2-\beta)+1]\Gamma_q(2-\gamma)}{bR\Gamma_q(2-\beta)}<1,$$

则边值问题(1)存在唯一解 u：$\|u\|\leqslant R$.

证明：设

$$U_R=\{u:\ u\in C([0,1],\mathbb{R}),\ {}^cD_q^\beta u\in C([0,1],\mathbb{R}),\ \|u\|\leqslant R\},$$

则对任意的 $t\in[0,1]$，$u\in U_R$，有

$$\begin{aligned}|(\Phi u)(t)|\leqslant{}&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs+\\&\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs+\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)b}+\\&\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs+\\&\frac{\lambda}{b}\Gamma_q(2-\gamma)+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs\leqslant\\&\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}\right][LR+\max_{t\in[0,1]}|f(t,0,0)|]+\\&\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)b}+\frac{\lambda}{b}\Gamma_q(2-\gamma).\end{aligned}$$

因为

$$\begin{aligned}|({}^cD_q^\beta(\Phi u))(t)|\leqslant{}&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs+\\&\frac{1}{\Gamma_q(2-\beta)}\left[\frac{\lambda}{b}\Gamma_q(2-\gamma)+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))|\,\mathrm{d}_qs\right]\leqslant\\&\left[\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right][LR+\max_{t\in[0,1]}|f(t,0,0)|]+\frac{\lambda\Gamma_q(2-\gamma)}{b\Gamma_q(2-\beta)},\end{aligned}$$

故 $\|\Phi u\|\leqslant R$，即 $\Phi:U_R\to U_R$.

对任意的 $t\in[0,1]$，$u,v\in U_R$，有

$$\begin{aligned}|(\Phi u)(t)-(\Phi v)(t)|\leqslant{}&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\,\mathrm{d}_qs+\\&\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\,\mathrm{d}_qs+\\&\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\,\mathrm{d}_qs+\\&\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1(1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\,\mathrm{d}_qs\leqslant\end{aligned}$$

$$\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} L(|u(s)-v(s)|+|{}^c(D_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_q s+$$

$$\frac{a}{(1+a)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} L(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_q s+$$

$$\frac{a\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma)}\int_0^1 (1-qs)^{(\alpha-\gamma-1)} L(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_q s+$$

$$\frac{\Gamma_q(2-\gamma)}{\Gamma_q(\alpha-\gamma)}\int_0^1 (1-qs)^{(\alpha-\gamma-1)} L(|u(s)-v(s)|+|({}^cD_q^\beta u)(s)-({}^cD_q^\beta v)(s)|)\mathrm{d}_q s\leqslant$$

$$L\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}\right]\|u-v\|.$$

因为

$$|({}^cD_q^\beta(\Phi u))(t)-({}^cD_q^\beta(\Phi v))(t)|\leqslant$$

$$\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t (t-qs)^{(\alpha-\beta-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_q s+$$

$$\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma)}\int_0^1 (1-qs)^{(\alpha-\gamma-1)}|f(s,u(s),({}^cD_q^\beta u)(s))-f(s,v(s),({}^cD_q^\beta v)(s))|\mathrm{d}_q s\leqslant$$

$$L\left[\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\|u-v\|,$$

所以

$$\|\Phi u-\Phi v\|\leqslant L\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\|u-v\|.$$

从而 Φ 是压缩映射，根据 Banach 不动点定理知，系统(1)存在唯一解.

3　应用实例

例 1　考虑边值问题

$$\begin{cases}({}^cD_{0.5}^{1.5}u)(t)=K\{\sin t+\arctan[u(t)+({}^cD_{0.5}^{0.75}u)(t)]\},\quad 0<t<1,\\ u(0)+2u(1)=0,\quad ({}^cD_{0.5}^{0.25}u)(0)+\sqrt{\pi}({}^cD_{0.5}^{0.25}u)(1)=\dfrac{1}{100},\end{cases}\tag{4}$$

其中 $q=0.5$，$\alpha=1.5$，$\beta=0.75$，$\gamma=0.25$，$a=2$，$b=\sqrt{\pi}$，$\lambda=\dfrac{1}{100}$，K 为常数.

设 $f(t,x,y)=K[\sin t+\arctan(x+y)]$，则式(2)成立. 若

$$K<\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]^{-1}=$$

$$\left[\frac{5}{3\Gamma_{0.5}(2.5)}+\frac{5\Gamma_{0.5}(1.75)}{3\Gamma_{0.5}(2.25)}+\frac{1}{\Gamma_{0.5}(1.75)}+\frac{\Gamma_{0.5}(1.75)}{\Gamma_{0.5}(1.25)\Gamma_{0.5}(2.25)}\right]^{-1},$$

则根据定理 1 知，边值问题(4)存在唯一解.

例 2　考虑反周期边值问题

$$\begin{cases}({}^cD_{0.5}^{1.5}u)(t)=\dfrac{1}{(\tan t+5)^2}\left\{\dfrac{t[(u(t))^2-(({}^cD_{0.5}^{0.5}u)(t))^2]}{1+[u(t)]^2+[({}^cD_{0.5}^{0.5}u)(t)]^2}+\dfrac{1}{2}(1+\cos t)\right\},\quad 0<t<1,\\ u(0)+u(1)=0,\quad ({}^cD_{0.5}^{0.25}u)(0)+({}^cD_{0.5}^{0.25}u)(1)=0,\end{cases}\tag{5}$$

其中 $q=0.5$，$\alpha=1.5$，$\beta=0.5$，$\gamma=0.25$，$a=1$，$b=1$，$\lambda=0$.

设 $f(t,x,y)=\dfrac{1}{(\tan t+5)^2}\left[\dfrac{t(x^2-y^2)}{1+x^2+y^2}+\dfrac{1}{2}(1+\cos t)\right]$，则 $\max\limits_{t\in[0,1]}|f(t,0,0)|=\dfrac{1}{25}$. 当 $L=\dfrac{1}{25}$，$R=1$ 时，式(3)成立. 于是

$$\theta=\left[\frac{1+2a}{(1+a)\Gamma_q(\alpha+1)}+\frac{(1+2a)\Gamma_q(2-\gamma)}{(1+a)\Gamma_q(\alpha-\gamma+1)}+\frac{1}{\Gamma_q(\alpha-\beta+1)}+\frac{\Gamma_q(2-\gamma)}{\Gamma_q(2-\beta)\Gamma_q(\alpha-\gamma+1)}\right]\times$$

$$\left[L+\frac{1}{R}\max_{t\in[0,1]}|f(t,0,0)|\right]+\frac{a\lambda\Gamma_q(2-\gamma)}{(1+a)bR}+\frac{\lambda[\Gamma_q(2-\beta)+1]\Gamma_q(2-\gamma)}{bR\Gamma_q(2-\beta)}=$$

$$\frac{2}{25}\left[\frac{3}{2\Gamma_{0.5}(2.5)}+\frac{3\Gamma_{0.5}(1.75)}{2\Gamma_{0.5}(2.25)}+\frac{1}{\Gamma_{0.5}(2)}+\frac{\Gamma_{0.5}(1.75)}{\Gamma_{0.5}(1.5)\Gamma_{0.5}(2.25)}\right]<1,$$

根据定理2知，边值问题(5)存在唯一解 u：$\|u\|\leqslant 1$.

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