

Existence and uniqueness of solutions for a class of fractional-order nonlinear damped problems with time delay

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Abstract The existence and uniqueness of solutions to a class of fractional-order nonlinear damped problems with time delay is studied. Several sufficient conditions are established to ensure the existence and uniqueness of solutions. The proofs depend mainly on Banach's contraction mapping principle, Krasnoselskii's fixed-point theorem, and a generalized Henry-type Gronwall inequality. The obtained results extend those conclusions in the existing literature.

Keywords fractional-order delayed equation; nonlinear damped equation; existence; uniqueness; fixed point theorem

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一类带时滞的分数阶非线性阻尼问题解的存在性与唯一性

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摘要 研究了一类带时滞的分数阶非线性阻尼问题解的存在性和唯一性. 利用 Banach 压缩映射原理、Krasnoselskii 不动点定理以及一个广义 Henry 型 Gronwall 不等式, 给出了几个保证解存在唯一性的充分条件. 所得结果推广了已有文献中的相关结论.

关键词 分数阶时滞方程; 非线性阻尼方程; 存在性; 唯一性; 不动点定理

Fractional-order calculus has garnered considerable attention due to its remarkable applicability in a wide range of fields^[1-4]. In contrast to integer-order calculus, fractional-order calculus inherently exhibits infinite memory and enhanced degrees of freedom. Therefore, it can more accurately characterize phenomena in many fields such as physics, economics, medicine, and biology.

Damping is a ubiquitous physical phenomenon in oscillatory or vibratory systems. Its effects cannot be ignored in the analysis of some structures that contain rubbers, polymers, or viscoelastic materials. Over the past several decades, various material damping

models have been formulated using fractional-order differential or integral equations. For certain damping materials, fractional calculus offers a particularly efficient modeling tool, yielding simpler representations that involve only a small number of parameters. In recent years, many scholars have established different types of fractional-order damped equations and have reported a wealth of interesting results regarding the properties of their solutions, including existence^[5-7], uniqueness^[5-6], controllability^[8-12], and stability^[13-14].

It is well known that the existence and uniqueness of solutions are the prerequisite for analyzing the dynamics of systems. In [6], Boulares et al. studied

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the existence and uniqueness of positive solutions for the following fractional differential equation:

$$\begin{cases} {}^C\mathcal{D}_t^{\gamma_2}h(t) = f(t, h(t)) + {}^C\mathcal{D}_t^{\gamma_2-1}g(t, h(t)), & t \in [0, T], \\ h(0) = \phi(0) > 0, \quad h'(0) = \phi'(0) > 0, \end{cases} \quad (1)$$

where $1 < \gamma_2 \leq 2$. The proof depends mainly on the method of upper and lower solutions, the Schauder fixed point theorem, and the Banach fixed point

$$\begin{cases} {}^C\mathcal{D}_t^{\gamma_2}h(t) - Q {}^C\mathcal{D}_t^{\gamma_1}h(t) = f(t, h(t)), & t \in [0, T], \\ h(0) = \phi(0), \quad h'(0) = \phi'(0), \end{cases} \quad (2)$$

where $0 < \gamma_1 \leq 1 < \gamma_2 \leq 2$. The proofs rely mainly on the Banach fixed point theorem, the Krasnoselskii fixed-point theorem, and the Schauder fixed point

$$\begin{cases} {}^C\mathcal{D}_t^{\gamma_1}h(t) = Q_0h(t) + Q_1h(t - \eta) + f(t, h(t), h(t - \eta)), & t \in [0, T], \\ h(t) = \phi(t), & t \in [-\eta, 0], \end{cases} \quad (3)$$

where $0 < \gamma_1 \leq 1$. The existence and uniqueness of the solution were proved by using the supremum technique and a generalized Gronwall inequality.

$$\begin{cases} {}^C\mathcal{D}_t^{\gamma_2}h(t) - Q {}^C\mathcal{D}_t^{\gamma_1}h(t) = f(t, h(t), h(t - \eta)), & t \in [0, T], \\ h(t) = \phi(t), \quad h'(t) = \phi'(t), & t \in [-\eta, 0], \end{cases} \quad (4)$$

where $0 < \gamma_1 \leq 1 < \gamma_2 \leq 2$; Q is a constant matrix in $\mathbb{R}^{n \times n}$, $\eta > 0$ is the time delay; $f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, and $\phi \in C^1([-\eta, 0], \mathbb{R}^n)$ is the given initial function. We establish several sufficient conditions to ensure the existence and uniqueness of the solution to Problem (4). The proofs depend mainly on Banach's contraction mapping principle, Krasnoselskii's fixed-point theorem, and a generalized Henry-type Gronwall inequality. It is worth pointing out that the assumption on f covers those in [5, 6, 15]. Our proof methods differ from those in [6, 15]. In particular, Theorems 1-2 in this paper extend those in [5]. In addition, we also establish an existence and uniqueness theorem (Theorem 3) using a generalized Henry-type Gronwall inequality with delay.

The rest of this paper is organized as follows. Section 1 recalls some necessary preliminaries. Section 2 establishes several theorems on the existence and uniqueness of solutions, along with a corollary. In Section 3, we provide an example to show the correctness of our results.

theorem. In [5], Sheng and Jiang considered the existence and uniqueness of the solution to the following damped system:

theorem. In [15], Wang et al. considered the following nonlinear fractional-order system with delay:

Motivated by the above-mentioned works, we are concerned with the following fractional nonlinear damped problem with time delay:

1 Preliminaries

This section presents some necessary notation and recalls several definitions as well as auxiliary lemmas.

Let $\mathbb{N}_0$ denote the set of nonnegative integers and $\mathbb{R}_+$ the set of nonnegative real numbers. Let $T > 0$ be a constant, and let $|x| = \max_{1 \leq i \leq n} |x_i|$ be the infinite norm in $\mathbb{R}^n$. Denote by $\mathcal{C}([a, b], \mathbb{R}^n)$ the Banach space of all continuous functions mapping $[a, b]$ into $\mathbb{R}^n$, equipped with the norm $\|h\|_c = \max_{t \in [a, b]} |h(t)|$. For an arbitrary matrix $Q = (a_{ij}) \in \mathbb{R}^{n \times n}$, let $\|Q\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$. For $1 \leq p < \infty$, $L^p([a, b], \mathbb{R}^n)$ denotes the Lebesgue space of p -integrable functions, endowed with the norm $\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p}$. $\mathcal{AC}([a, b], \mathbb{R}^n)$ denotes the space of absolutely continuous functions on $[a, b]$, and $\mathcal{AC}^m([a, b], \mathbb{R}^n)$ denotes the space of functions $f \in \mathcal{C}^{m-1}([a, b], \mathbb{R}^n)$ with $f^{(m-1)} \in \mathcal{AC}([a, b], \mathbb{R}^n)$, where $m \geq 2$ is an integer.

Definition 1^[16]. Let $\alpha \in \mathbb{R}^+$ and $f \in L^1([0, \infty), \mathbb{R}^n)$, the Riemann-Liouville fractional integral of order α is

defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2^[16]. Let $[\alpha]$ denote the integer part of $\alpha \in \mathbb{R}^+$. Let $m = [\alpha] + 1$. For $f \in \mathcal{AC}^m([0, \infty), \mathbb{R}^n)$, the Riemann-Liouville fractional derivative of order α is defined by

$${}_0^L \mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \quad t > 0.$$

Definition 3^[17]. For $\alpha \in \mathbb{R}^+$, let $m = [\alpha] + 1$ when $\alpha \notin \mathbb{N}_0$. If $f \in \mathcal{AC}^m([0, \infty), \mathbb{R}^n)$, then the Caputo fractional derivative of order α is defined by

$${}_0^C \mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad t > 0.$$

Based on Definitions 2-3, we have

$${}_0^C \mathcal{D}_t^\alpha f(t) = {}_0^L \mathcal{D}_t^\alpha \left[f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0) \right].$$

In particular, when $\alpha \in \mathbb{N}_0$, we have ${}_0^C \mathcal{D}_t^\alpha f(t) = f^{(\alpha)}(t)$.

Lemma 1 (Krasnoselskii's Fixed-Point Theorem^[18]). Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $(\mathbb{X}, \|\cdot\|)$. Suppose that two operators $\mathcal{P}_1, \mathcal{P}_2: \Omega \rightarrow \mathbb{X}$ satisfy:

(i) $\mathcal{P}_1 x + \mathcal{P}_2 y \in \Omega$ for every $x, y \in \Omega$;

$$\begin{cases} w(t) \leq \mathfrak{h}(t) + 1(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\mu_1(s)w(s) + \kappa_1(s)w(s-\tau)] ds \\ \quad + \mathfrak{Z}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} [\mu_2(s)w(s) + \kappa_2(s)w(s-\tau)] ds, & t \in [0, T], \\ w(t) \leq \varpi(t), & t \in [-\tau, 0], \end{cases} \quad (5)$$

then

$$w(t) \leq P^{1/q}(t) \exp\{G(t)2t/q\}, \quad t \in [0, T],$$

where

$$Z(t) = 3^{q-1} \left\{ \frac{1(t)}{\Gamma(\alpha)} \frac{t^{\alpha-1/q} \mu_1(t)}{((\alpha-1)p+1)^{1/p}} + \frac{\mathfrak{Z}(t)}{\Gamma(\beta)} \frac{t^{\beta-1/q} \mu_2(t)}{((\beta-1)p+1)^{1/p}} \right\}^q,$$

$$R(t) = 3^{q-1} \left\{ \frac{1(t)}{\Gamma(\alpha)} \frac{t^{\alpha-1/q} \kappa_1(t)}{((\alpha-1)p+1)^{1/p}} + \frac{\mathfrak{Z}(t)}{\Gamma(\beta)} \frac{t^{\beta-1/q} \kappa_2(t)}{((\beta-1)p+1)^{1/p}} \right\}^q,$$

$$G(t) = \max\{Z(t), R(t)\}, \quad P(t) = 3^{q-1} \mathfrak{h}^q(t),$$

$p, q > 0$, $1/p + 1/q = 1$, $\alpha > 1/q$, and $\beta > 1/q$.

(ii) $\mathcal{P}_1$ is continuous and compact;

(iii) $\mathcal{P}_2$ is a contraction with constant $\kappa < 1$.

Then there exists a point $\omega \in \Omega$ such that $\mathcal{P}_1 \omega + \mathcal{P}_2 \omega = \omega$.

Lemma 2 (Arzelà-Ascoli Theorem^[19]). Let K be a compact metric space and let $\mathcal{F} \subset \mathcal{C}(K, \mathbb{R}^n)$. If $\mathcal{F}$ is uniformly bounded and equicontinuous on K , then $\mathcal{F}$ is relatively compact in the Banach space $(\mathcal{C}(K, \mathbb{R}^n), \|\cdot\|_c)$.

Lemma 3 (Banach Fixed-Point Theorem^[20]). Let (X, d) be a complete metric space and let $\mathcal{P}: X \rightarrow X$ be a contraction, i.e., there exists $k \in [0, 1)$ such that

$$d(\mathcal{P}x, \mathcal{P}y) \leq k d(x, y) \quad \text{for all } x, y \in X.$$

Then $\mathcal{P}$ admits a unique fixed point in X .

Lemma 4 (Hölder's Inequality^[21]). Let $p, q \geq 1$ be conjugate exponents satisfying $1/p + 1/q = 1$. If $\phi \in L^p([a, b])$ and $\varphi \in L^q([a, b])$, then the product $\phi\varphi$ belongs to $L^1([a, b])$ and

$$\|\phi\varphi\|_{L^1([a, b])} \leq \|\phi\|_{L^p([a, b])} \|\varphi\|_{L^q([a, b])}.$$

Lemma 5^[22]. Let $\alpha, \beta > 0$, $\mathfrak{h}, \ell, \mathfrak{Z}, \mu_1, \mu_2, \kappa_1, \kappa_2 \in \mathcal{C}([0, T], \mathbb{R}_+)$, and $\varpi(t) \in \mathcal{C}([-\tau, 0], \mathbb{R}_+)$. Let $\mathfrak{h}(0) = \varpi(0)$. Suppose that $\mathfrak{h}, \ell, \mathfrak{Z}, \mu_1, \mu_2, \kappa_1, \kappa_2$ are nondecreasing functions. If $w \in \mathcal{C}([-\tau, T], \mathbb{R}_+)$ satisfies

2 Main results

This section focuses on the existence and uniqueness of solutions to Problem (4). Let us first present some assumptions on the nonlinear function f .

(H1) The function $f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous.

(H2) There exists a non-negative function $\delta \in L^{\rho_1^*}([0, T], \mathbb{R}^+)$ such that

$$|f(t, \alpha, \beta)| \leq \delta(t),$$

for all $t \in [0, T]$ and $\alpha, \beta \in \mathbb{R}^n$,

where $\rho_1^* > 1$ is a given constant.

(H3) There exist two functions $\theta_1 \in L^{\rho_2^*}([0, T], \mathbb{R}^+)$ and $\theta_2 \in L^{\rho_3^*}([0, T], \mathbb{R}^+)$ such that

$$|f(t, \alpha_1, \beta_1) - f(t, \alpha_2, \beta_2)| \leq \theta_1(t)|\alpha_1 - \alpha_2| + \theta_2(t)|\beta_1 - \beta_2|,$$

$$h(t) = \begin{cases} \phi(t), & t \in [-\eta, 0], \\ \phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q \phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} \\ \quad + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} h(s) ds \\ \quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} f(s, h(s), h(s-\eta)) ds, & t \in (0, T]. \end{cases} \quad (6)$$

Lemma 6. The set $\mathcal{C}_\omega$ is a non-empty, closed, and convex subset of $\mathcal{C}([-\eta, T], \mathbb{R}^n)$.

Proof. It is clear that $\mathcal{C}_\omega$ is not empty. Indeed, consider the following function z defined by

$$z(t) = \begin{cases} \phi(t), & t \in [-\eta, 0], \\ \phi(0), & t \in (0, T]. \end{cases}$$

Obviously, it is continuous on $[-\eta, T]$ since ϕ is continuous on $[-\eta, 0]$. Moreover,

$$\|z\|_c = \max\{\|\phi\|_{\mathcal{C}([-\eta, 0])}, |\phi(0)|\} \leq \omega.$$

Thus, $z \in \mathcal{C}_\omega$.

Next, we prove that $\mathcal{C}_\omega$ is convex. For $z_1, z_2 \in \mathcal{C}_\omega$ and $\theta \in [0, 1]$, put $w = \theta z_1 + (1 - \theta) z_2$. For any $t \in [-\eta, 0]$, we have

$$w(t) = \theta \phi(t) + (1 - \theta) \phi(t) = \phi(t).$$

Moreover, the triangle inequality allows us to obtain $\|w\|_c \leq \theta \|z_1\|_c + (1 - \theta) \|z_2\|_c \leq \theta \omega + (1 - \theta) \omega = \omega$. Consequently, $w \in \mathcal{C}_\omega$. This indicates that $\mathcal{C}_\omega$ is convex.

for all $t \in [0, T]$ and all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^n$, where $\rho_2^*, \rho_3^* > 1$ are two given constants.

For convenience in the subsequent analysis, let ρ_1, ρ_2 , and ρ_3 be the conjugate exponents corresponding to ρ_1^*, ρ_2^* , and ρ_3^* in (H2) and (H3), respectively. That is,

$$\rho_k = \frac{\rho_k^*}{\rho_k^* - 1}, \quad k = 1, 2, 3.$$

Define $\mathcal{C}_\omega$ by

$$\mathcal{C}_\omega = \left\{ h \in \mathcal{C}([-\eta, T], \mathbb{R}^n) : h(t) = \phi(t) \text{ for } t \in [-\eta, 0], \|h\|_c \leq \omega \right\},$$

where ω is a constant such that $\omega \geq \|\phi\|_{\mathcal{C}([-\eta, 0], \mathbb{R}^n)}$.

By Lemma 3.1 in [7], a function $h \in \mathcal{C}([-\eta, T], \mathbb{R}^n)$ is a solution of Problem (4) if and only if it satisfies the following Volterra-type integral equation

$$t \in [-\eta, 0],$$

It remains to show that $\mathcal{C}_\omega$ is closed. Let $\{z^{(k)}\} \subset \mathcal{C}_\omega$ be a sequence such that $z^{(k)} \rightarrow z$ in $\mathcal{C}([-\eta, T], \mathbb{R}^n)$. It is easy to see that

$$z(t) = \lim_{k \rightarrow \infty} z^{(k)}(t) = \phi(t), \quad \forall t \in [-\eta, 0].$$

Additionally, the continuity of the norm implies

$$\|z\|_c = \lim_{k \rightarrow \infty} \|z^{(k)}\|_c \leq \omega.$$

Therefore, $z \in \mathcal{C}_\omega$. We conclude that $\mathcal{C}_\omega$ is a closed subset of $\mathcal{C}([-\eta, T], \mathbb{R}^n)$.

Theorem 1. Suppose that (H1)-(H3) hold. If

$$\frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} + \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} < 1, \quad (7)$$

then Problem (4) has a unique solution on $[-\eta, T]$.

Proof. Notice that the condition (7) evidently implies

$$\frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} < 1. \quad \text{This allows us to choose a}$$

constant Q such that

$$Q \geq \frac{\Gamma(\gamma_2 - \gamma_1 + 1)}{\Gamma(\gamma_2 - \gamma_1 + 1) - \|Q\| T^{\gamma_2 - \gamma_1}} \left(\|\phi\|_{\mathcal{C}([- \eta, 0], \mathbb{R}^n)} + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} |\phi(0)| + \frac{T^{\gamma_2 - 1/\rho_1^*} \|\delta\|_{L^{\rho_1^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_1 + 1)^{1/\rho_1}} \right).$$

Let

$\|\cdot\|_c$ is a complete metric space.

$\mathcal{C}_Q = \{h \in \mathcal{C}([- \eta, T], \mathbb{R}^n) : h(t) = \phi(t) \text{ for } t \in [- \eta, 0], \|h\|_c \leq Q\}$.

The operator $F: \mathcal{C}_Q \rightarrow \mathcal{C}([- \eta, T], \mathbb{R}^n)$ is defined

as

By Lemma 6, $\mathcal{C}_Q$ equipped with the supremum norm

$$(Fh)(t) = \begin{cases} \phi(t), & t \in [- \eta, 0], \\ \phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q\phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} \\ \quad + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} h(s) ds \\ \quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} f(s, h(s), h(s - \eta)) ds, & t \in (0, T]. \end{cases}$$

First, let us prove $F(\mathcal{C}_Q) \subseteq \mathcal{C}_Q$. Let $h \in \mathcal{C}_Q$.

Q. For $t \in (0, T]$, by (H2), Hölder's inequality and

Evidently, $Fh \in \mathcal{C}([- \eta, T], \mathbb{R}^n)$. For $t \in [- \eta, 0]$,

$\|h\|_c \leq Q$, we derive

it is obvious that $|(Fh)(t)| = |\phi(t)| \leq \|\phi\|_{\mathcal{C}([- \eta, 0], \mathbb{R}^n)} \leq$

$$\begin{aligned} |(Fh)(t)| &= |\phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q\phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} f(s, h(s), h(s - \eta)) ds| \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} |\phi(0)| \\ &\quad + \frac{\|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} |h(s)| ds + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} |\delta(s)| ds \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} |\phi(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} Q \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \left(\int_0^t (t-s)^{(\gamma_2 - 1)\rho_1} ds \right)^{1/\rho_1} \|\delta\|_{L^{\rho_1^*}} \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} (|\phi(0)| + Q) + \frac{T^{\gamma_2 - 1/\rho_1^*} \|\delta\|_{L^{\rho_1^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_1 + 1)^{1/\rho_1}} \\ &\leq Q. \end{aligned}$$

This indicates $\|Fh\|_c \leq Q$. Hence, F maps $\mathcal{C}_Q$ into $\mathcal{C}_Q$.

evident that $|(Fh_1)(t) - (Fh_2)(t)| = 0$. For $t \in (0, T]$,

Now, let us show that F is a contraction mapping.

we apply (H3) to get

Suppose that $h_1, h_2 \in \mathcal{C}_Q$. When $t \in [- \eta, 0]$, it is

$$\begin{aligned}
& |(Fh_1)(t) - (Fh_2)(t)| \\
& \leq \frac{\|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} |h_1(s) - h_2(s)| ds \\
& + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} |f(s, h_1(s), h_1(s-\eta)) - f(s, h_2(s), h_2(s-\eta))| ds \\
& \leq \frac{\|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} |h_1(s) - h_2(s)| ds \\
& + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} \theta_1(s) |h_1(s) - h_2(s)| ds \\
& + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} \theta_2(s) |h_1(s-\eta) - h_2(s-\eta)| ds.
\end{aligned}$$

Applying Hölder's inequality to the integrals involving θ_1 and θ_2 , we obtain

$$\begin{aligned}
& |(Fh_1)(t) - (Fh_2)(t)| \\
& \leq \frac{\|Q\| \|h_1 - h_2\|_c}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} ds \\
& + \frac{\|h_1 - h_2\|_c}{\Gamma(\gamma_2)} \left(\int_0^t (t-s)^{\gamma_2 - 1} \theta_1(s) ds + \int_0^t (t-s)^{\gamma_2 - 1} \theta_2(s) ds \right) \\
& \leq \|h_1 - h_2\|_c \left[\frac{\|Q\| t^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \right. \\
& \quad \left. + \frac{1}{\Gamma(\gamma_2)} \left(\left(\int_0^t (t-s)^{(\gamma_2 - 1)\rho_2} ds \right)^{1/\rho_2} \|\theta_1\|_{L^{\rho_2}} + \left(\int_0^t (t-s)^{(\gamma_2 - 1)\rho_3} ds \right)^{1/\rho_3} \|\theta_2\|_{L^{\rho_3}} \right) \right] \\
& \leq \|h_1 - h_2\|_c \left[\frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} + \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|(Fh_1) - (Fh_2)\|_c \\
& \leq \|h_1 - h_2\|_c \left[\frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} + \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} \right].
\end{aligned}$$

Combining this with the condition (7), we conclude and

that F is a contraction mapping on the complete metric space $\mathcal{C}_\eta$. According to Banach's contraction mapping principle, F has a unique fixed point $h^* \in \mathcal{C}_\eta$, which is the unique solution to Problem (4) on $[-\eta, T]$. This completes the proof.

Remark 1. When the time delay η is absent in Problem (4), Theorem 1 reduces to Theorem 4.1 in [5].

Theorem 2. Suppose that (H1)-(H3) hold. If

$$\frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} < 1, \quad (8)$$

$$\begin{aligned}
& \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} \\
& + \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} < 1, \quad (9)
\end{aligned}$$

then Problem (4) admits at least one solution on $[-\eta, T]$.

Proof. By (8), we can choose a constant q such that

$$\varrho \geq \frac{\Gamma(\gamma_2 - \gamma_1 + 1)}{\Gamma(\gamma_2 - \gamma_1 + 1) - \|Q\| T^{\gamma_2 - \gamma_1}} \left(\|\phi\|_{\mathcal{C}([- \eta, 0], \mathbb{R}^n)} + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} |\phi(0)| + \frac{T^{\gamma_2 - 1/\rho_1^*} \|\delta\|_{L^{\rho_1^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_1 + 1)^{1/\rho_1}} \right).$$

Let

$$\mathcal{C}_\varrho = \left\{ h \in \mathcal{C}([- \eta, T], \mathbb{R}^n) : h(t) = \phi(t) \text{ for } t \in [- \eta, 0], \|h\|_\varrho \leq \varrho \right\}.$$

By Lemma 6, $\mathcal{C}_\varrho$ is a nonempty, bounded, closed, and convex subset of $\mathcal{C}([- \eta, T], \mathbb{R}^n)$. Define the operators $F_1, F_2: \mathcal{C}_\varrho \rightarrow \mathcal{C}([- \eta, T], \mathbb{R}^n)$ by

$$(F_1 h)(t) = \begin{cases} \phi(t), & t \in [- \eta, 0], \\ \phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q \phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} h(s) ds, & t \in (0, T], \end{cases}$$

and

$$(F_2 h)(t) = \begin{cases} 0, & t \in [- \eta, 0], \\ \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} f(s, h(s), h(s-\eta)) ds, & t \in (0, T]. \end{cases}$$

We first verify $F_1 x + F_2 h \in \mathcal{C}_\varrho$ for all $x, h \in \mathcal{C}_\varrho$. Clearly, $F_1 x + F_2 h$ is continuous. For any $t \in [- \eta, 0]$, $(F_1 x + F_2 h)(t) = \phi(t)$. Moreover, we get that

$|(F_1 x + F_2 h)(t)| = |\phi(t)| \leq \|\phi\|_{\mathcal{C}([- \eta, 0], \mathbb{R}^n)} \leq \varrho$. For $t \in (0, T]$, the use of (H2) and Hölder's inequality yield

$$\begin{aligned} & |(F_1 x)(t) + (F_2 h)(t)| \\ &= |\phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q \phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} x(s) ds \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} f(s, h(s), h(s-\eta)) ds| \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} |\phi(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \varrho \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2 - 1} \delta(s) ds \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{T^{\gamma_2 - \gamma_1} \|Q\| (|\phi(0)| + \varrho)}{\Gamma(\gamma_2 - \gamma_1 + 1)} \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \left(\int_0^t (t-s)^{(\gamma_2 - 1)\rho_1} ds \right)^{1/\rho_1} \left(\int_0^t \delta(s)^{\rho_1^*} ds \right)^{1/\rho_1^*} \\ &\leq |\phi(0)| + T|\phi'(0)| + \frac{T^{\gamma_2 - \gamma_1} \|Q\| (|\phi(0)| + \varrho)}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{T^{\gamma_2 - 1/\rho_1^*} \|\delta\|_{L^{\rho_1^*}}}{\Gamma(\gamma_2) ((\gamma_2 - 1)\rho_1 + 1)^{1/\rho_1}} \\ &\leq \varrho. \end{aligned}$$

This implies $\|F_1 x + F_2 h\|_\varrho \leq \varrho$. Hence, $F_1 x + F_2 h \in \mathcal{C}_\varrho$ for all $x, h \in \mathcal{C}_\varrho$.

mapping. Let $h_1, h_2 \in \mathcal{C}_\varrho$. For $t \in [- \eta, 0]$, $(F_2 h_1)(t) - (F_2 h_2)(t) = 0$. For $t \in (0, T]$, using (H3), we obtain

Next, we demonstrate that F_2 is a contraction

$$\begin{aligned}
& |(F_2 h_1)(t) - (F_2 h_2)(t)| \\
&= \left| \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} f(s, h_1(s), h_1(s-\eta)) ds - \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} f(s, h_2(s), h_2(s-\eta)) ds \right| \\
&\leq \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} [\theta_1(s) |h_1(s) - h_2(s)| + \theta_2(s) |h_1(s-\eta) - h_2(s-\eta)|] ds
\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
|(F_2 h_1)(t) - (F_2 h_2)(t)| &\leq \frac{\|h_1 - h_2\|_c}{\Gamma(\gamma_2)} \left[\int_0^t (t-s)^{\gamma_2-1} \theta_1(s) ds + \int_0^t (t-s)^{\gamma_2-1} \theta_2(s) ds \right] \\
&\leq \frac{\|h_1 - h_2\|_c}{\Gamma(\gamma_2)} \left[\left(\int_0^t (t-s)^{(\gamma_2-1)\rho_2} ds \right)^{1/\rho_2} \left(\int_0^t \theta_1(s)^{\rho_2^*} ds \right)^{1/\rho_2^*} \right. \\
&\quad \left. + \left(\int_0^t (t-s)^{(\gamma_2-1)\rho_3} ds \right)^{1/\rho_3} \left(\int_0^t \theta_2(s)^{\rho_3^*} ds \right)^{1/\rho_3^*} \right] \\
&\leq \|h_1 - h_2\|_c \left[\frac{T^{\gamma_2-1/\rho_2^*} \|\theta_1\|_{L^{\rho_2^*}}}{\Gamma(\gamma_2) ((\gamma_2-1)\rho_2 + 1)^{1/\rho_2}} + \frac{T^{\gamma_2-1/\rho_3^*} \|\theta_2\|_{L^{\rho_3^*}}}{\Gamma(\gamma_2) ((\gamma_2-1)\rho_3 + 1)^{1/\rho_3}} \right].
\end{aligned}$$

Therefore,

$$\|(F_2 h_1) - (F_2 h_2)\|_c \leq \left[\frac{T^{\gamma_2-1/\rho_2^*} \|\theta_1\|_{L^{\rho_2^*}}}{\Gamma(\gamma_2) ((\gamma_2-1)\rho_2 + 1)^{1/\rho_2}} + \frac{T^{\gamma_2-1/\rho_3^*} \|\theta_2\|_{L^{\rho_3^*}}}{\Gamma(\gamma_2) ((\gamma_2-1)\rho_3 + 1)^{1/\rho_3}} \right] \|h_1 - h_2\|_c.$$

With the condition (9), F_2 is a contraction mapping.

In what follows, we will prove that the operator F_1 is continuous and compact.

that $\{h_n\}_{n=1}^\infty \subset \mathcal{C}_q$ is a sequence such that $h_n \rightarrow h$ uniformly on $[-\eta, T]$. For any $t \in [-\eta, 0]$, $(F_1 h_n - F_1 h)(t) = 0$. For any $t \in [0, T]$, we obtain

Let us first discuss the continuity of F_1 . Suppose

$$\begin{aligned}
& |(F_1 h_n)(t) - (F_1 h)(t)| \\
&= \left| \phi(0) + t\phi'(0) - \frac{t^{\gamma_2-\gamma_1} Q\phi(0)}{\Gamma(\gamma_2-\gamma_1+1)} + \frac{Q}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} h_n(s) ds \right. \\
&\quad \left. - \left(\phi(0) + t\phi'(0) - \frac{t^{\gamma_2-\gamma_1} Q\phi(0)}{\Gamma(\gamma_2-\gamma_1+1)} + \frac{Q}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} h(s) ds \right) \right| \\
&= \left| \frac{Q}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} (h_n(s) - h(s)) ds \right| \\
&\leq \frac{\|Q\|}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} |h_n(s) - h(s)| ds \\
&\leq \frac{\|Q\| T^{\gamma_2-\gamma_1}}{\Gamma(\gamma_2-\gamma_1+1)} \|h_n - h\|_c.
\end{aligned}$$

Hence,

$$\|F_1 h_n - F_1 h\|_c \leq \frac{\|Q\| T^{\gamma_2-\gamma_1}}{\Gamma(\gamma_2-\gamma_1+1)} \|h_n - h\|_c.$$

Therefore, we obtain that $\|F_1 h_n - F_1 h\|_c$ converges to 0 in $\mathcal{C}([-\eta, T], \mathbb{R}^n)$. This implies the continuity

of F_1 .

In order to prove the compactness of $F_1(\mathcal{C}_q)$, we first consider the uniform boundedness of $F_1(\mathcal{C}_q)$. Let $h \in \mathcal{C}_q$. For $t \in [-\eta, 0]$, $|(F_1 h)(t)| \leq \|\phi\|_{\mathcal{C}([-\eta, 0])} \leq q$.

For $t \in (0, T]$, we obtain

$$\begin{aligned}
|(F_1 h)(t)| &= \left| \phi(0) + t\phi'(0) - \frac{t^{\gamma_2 - \gamma_1} Q \phi(0)}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{Q}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} h(s) ds \right| \\
&\leq |\phi(0)| + T|\phi'(0)| + \frac{T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \|Q\| |\phi(0)| + \frac{\|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \int_0^t (t-s)^{\gamma_2 - \gamma_1 - 1} |h(s)| ds \\
&\leq |\phi(0)| + T|\phi'(0)| + \frac{T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \|Q\| |\phi(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \varrho.
\end{aligned}$$

Thus,

$$\|F_1 h\|_c \leq |\phi(0)| + T|\phi'(0)| + \frac{T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \|Q\| |\phi(0)| + \frac{\|Q\| T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} \varrho.$$

This gives that, $F_1(\mathcal{C}_\varrho)$ is uniformly bounded. $[-\eta, T]$. On one hand, for $0 \leq t_1 < t_2 \leq T$, we have

Now we prove that $F_1(\mathcal{C}_\varrho)$ is equicontinuous on

$$\begin{aligned}
& |(F_1 h)(t_2) - (F_1 h)(t_1)| \\
& \leq |\phi'(0)|(t_2 - t_1) + \frac{|\phi(0)| \|Q\|}{\Gamma(\gamma_2 - \gamma_1 + 1)} (t_2^{\gamma_2 - \gamma_1} - t_1^{\gamma_2 - \gamma_1}) \\
& \quad + \frac{\|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \left| \int_0^{t_2} (t_2 - s)^{\gamma_2 - \gamma_1 - 1} h(s) ds - \int_0^{t_1} (t_1 - s)^{\gamma_2 - \gamma_1 - 1} h(s) ds \right| \\
& \leq |\phi'(0)|(t_2 - t_1) + \frac{|\phi(0)| \|Q\|}{\Gamma(\gamma_2 - \gamma_1 + 1)} (t_2^{\gamma_2 - \gamma_1} - t_1^{\gamma_2 - \gamma_1}) \\
& \quad + \frac{\varrho \|Q\|}{\Gamma(\gamma_2 - \gamma_1)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\gamma_2 - \gamma_1 - 1} ds + \int_0^{t_1} |(t_2 - s)^{\gamma_2 - \gamma_1 - 1} - (t_1 - s)^{\gamma_2 - \gamma_1 - 1}| ds \right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
& |(F_1 h)(t_2) - (F_1 h)(t_1)| \\
& \leq |\phi'(0)|(t_2 - t_1) + \frac{|\phi(0)| \|Q\|}{\Gamma(\gamma_2 - \gamma_1 + 1)} |t_2^{\gamma_2 - \gamma_1} - t_1^{\gamma_2 - \gamma_1}| \\
& \quad + \frac{\varrho \|Q\|}{\Gamma(\gamma_2 - \gamma_1 + 1)} \left[2(t_2 - t_1)^{\gamma_2 - \gamma_1} + |t_2^{\gamma_2 - \gamma_1} - t_1^{\gamma_2 - \gamma_1}| \right].
\end{aligned}$$

Obviously, the term on the right hand side of this inequality is irrelevant to h . As t_2 tends to t_1 , we have $|(F_1 h)(t_2) - (F_1 h)(t_1)| \rightarrow 0$. As a result, $F_1(\mathcal{C}_\varrho)$ is equicontinuous on $[0, T]$.

On the other hand, for $t \in [-\eta, 0]$, $(F_1 h)(t) = \phi(t)$. It is evident that $F_1(\mathcal{C}_\varrho)$ is equicontinuous on $[-\eta, 0]$. Moreover, $F_1(\mathcal{C}_\varrho)$ is equicontinuous on $[-\eta, T]$. Combining this with the uniform boundedness of $F_1(\mathcal{C}_\varrho)$, we use the Arzelà-Ascoli theorem to show that F_1 is a compact operator.

With the help of Krasnoselskii's fixed point theorem, we conclude that there exists $h^* \in \mathcal{C}_\varrho$ such that $F_1 h^* + F_2 h^* = h^*$. Consequently, h^* is a solution

to Problem (4) on $[-\eta, T]$. The proof is complete.

Remark 2. When the time delay η is absent in Problem (4), Theorem 2 reduces to Theorem 4.2 in [5].

To further investigate the uniqueness of the solution to Problem (4), we introduce the following assumption, which is stronger than (H3).

(H4) There exist two nondecreasing functions

$\theta_1, \theta_2 \in \mathcal{C}([0, T], \mathbb{R}^+)$ such that

$$\begin{aligned}
|f(t, \alpha_1, \beta_1) - f(t, \alpha_2, \beta_2)| &\leq \theta_1(t) |\alpha_1 - \alpha_2| \\
&\quad + \theta_2(t) |\beta_1 - \beta_2|,
\end{aligned}$$

for all $t \in [0, T]$ and all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^n$.

Theorem 3. Suppose that (H1), (H2) and (H4) hold. If

$$\frac{\|Q\|T^{\gamma_2-\gamma_1}}{\Gamma(\gamma_2-\gamma_1+1)} < 1,$$

and

$$\frac{T^{\gamma_2-1/\rho_2^*}}{\Gamma(\gamma_2)((\gamma_2-1)\rho_2+1)^{1/\rho_2}} + \frac{T^{\gamma_2-1/\rho_3^*}}{\Gamma(\gamma_2)((\gamma_2-1)\rho_3+1)^{1/\rho_3}} < 1,$$

$$h_i(t) = \phi(0) + t\phi'(0) - \frac{t^{\gamma_2-\gamma_1}Q\phi(0)}{\Gamma(\gamma_2-\gamma_1+1)} + \frac{Q}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} h_i(s) ds + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} f(s, h_i(s), h_i(s-\eta)) ds,$$

where $i = 1, 2$.

$$\begin{aligned} |\xi(t)| &= \left| \frac{Q}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} [h_1(s) - h_2(s)] ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} [f(s, h_1(s), h_1(s-\eta)) - f(s, h_2(s), h_2(s-\eta))] ds \right| \\ &\leq \frac{\|Q\|}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} |h_1(s) - h_2(s)| ds \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} |f(s, h_1(s), h_1(s-\eta)) - f(s, h_2(s), h_2(s-\eta))| ds \\ &\leq \frac{\|Q\|}{\Gamma(\gamma_2-\gamma_1)} \int_0^t (t-s)^{\gamma_2-\gamma_1-1} |\xi(s)| ds \\ &\quad + \frac{1}{\Gamma(\gamma_2)} \int_0^t (t-s)^{\gamma_2-1} [\theta_1(s)|\xi(s)| + \theta_2(s)|\xi(s-\eta)|] ds. \end{aligned}$$

The condition $0 < \gamma_1 \leq 1 < \gamma_2 \leq 2$ implies $\gamma_2 - \gamma_1 > 0$ and $\gamma_2 > 0$. Applying Lemma 5, we get

$$|\xi(t)| \leq 0, \quad t \in [0, T].$$

Hence, $|\xi(t)| = 0$ for any $t \in [-\eta, T]$.

$$\begin{cases} {}^C_0\mathcal{D}_t^{\gamma_2} h(t) - Q {}^C_0\mathcal{D}_t^{\gamma_1} h(t) = f(t, h(t)), & t \in [0, T], \\ h(0) = \phi(0), h'(0) = \phi'(0), \end{cases} \quad (10)$$

where $0 < \gamma_1 \leq 1 < \gamma_2 \leq 2$.

In this case, (H1), (H2) and (H4) reduce to the following assumptions.

(H1)' The function $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous.

(H2)' There exists a non-negative function $\delta \in L^{\rho_1^*}([0, T], \mathbb{R}^+)$ such that

$$|f(t, \alpha)| \leq \delta(t), \quad \text{for all } t \in [0, T] \text{ and } \alpha \in \mathbb{R}^n,$$

where $\rho_1^* > 1$ is a given constant.

(H4)' There exists $\theta \in \mathcal{C}([0, T], \mathbb{R}^+)$ such that

$$|f(t, \alpha_1) - f(t, \alpha_2)| \leq \theta(t)|\alpha_1 - \alpha_2|$$

then Problem (4) has a unique solution on $[-\eta, T]$.

Proof. According to Theorem 2, there exists at least one solution to Problem (4). Let h_1 and h_2 be two solutions to Problem (4), and let $\xi(t) = |h_1(t) - h_2(t)|$.

For $t \in [-\eta, 0]$, $h_1 = h_2 = \phi(t)$. For $t \in [0, T]$, it is obvious that h_1 and h_2 satisfy the equivalent integral equation (6). Namely,

Clearly, $\xi(t) = 0$, $t \in [-\eta, 0]$. For $t \in [0, T]$,

Moreover, $h_1(t) \equiv h_2(t)$ for $t \in [-\eta, T]$. This completes the proof of Theorem 3. $\square$

In the absence of time delay, Problem (4) reduces to the following form

for all $t \in [0, T]$ and all $\alpha_1, \alpha_2 \in \mathbb{R}^n$.

Corollary 1. Suppose that (H1)', (H2)' and (H4)' hold. If

$$\frac{\|Q\|T^{\gamma_2-\gamma_1}}{\Gamma(\gamma_2-\gamma_1+1)} < 1 \quad \text{and} \quad \frac{\|\theta\|_\infty T^{\gamma_2}}{\Gamma(\gamma_2+1)} < 1, \quad (11)$$

then Problem (10) has a unique solution on $[0, T]$.

3 Application

Example 1. For Problem (4), the parameters are

given as follows: $\gamma_1 = 1/2$, $\gamma_2 = 3/2$, $Q = 0.2$, $T = 1$,
 $\eta = 0.2$, $f(t, x, y) = \frac{1}{10} + \frac{1}{50} \sin x + \frac{1}{50} \sin y$, $\phi(t) =$

$1 + \frac{t}{2}$, $\phi'(t) = \frac{1}{2}$. The corresponding form is written as:

$$\begin{cases} {}^C_0 \mathcal{D}_t^{\gamma_2} h(t) - 0.2 {}^C_0 \mathcal{D}_t^{\gamma_2} h(t) = \frac{1}{10} + \frac{1}{50} \sin h(t) + \frac{1}{50} \sin h(t - 0.2), & 0 < t \leq 1, \\ h(t) = 1 + \frac{t}{2}, \quad h'(t) = \frac{1}{2} & -0.2 \leq t \leq 0. \end{cases} \quad (12)$$

Obviously, f is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and satisfies the following conditions:

$$|f(t, x, y)| \leq \frac{7}{50},$$

and

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \frac{1}{50} |x_1 - x_2| + \frac{1}{50} |y_1 - y_2|.$$

Let $\delta(t) = 7/50$ and $\theta_1(t) = \theta_2(t) = 1/50$, and let $\rho_1^* = \rho_2^* = \rho_3^* = \rho_2 = \rho_3 = 2$. Obviously, $\delta \in L^2([0, T], \mathbb{R}^+)$, $\theta_1 \in L^2([0, T], \mathbb{R}^+)$, and $\theta_2 \in L^2([0, T], \mathbb{R}^+)$.

(1) With the above parameters, we derive

$$\frac{|Q|T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} = 0.2 < 1,$$

and

$$\begin{aligned} & \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2^*}}}{\Gamma(\gamma_2)((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} + \\ & \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3^*}}}{\Gamma(\gamma_2)((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} = 2 \cdot \frac{1/50}{\Gamma(3/2)\sqrt{2}} \approx \\ & 0.0319 < 1. \end{aligned}$$

According to Theorem 2, Problem (12) has at least one solution on $[-0.2, 1]$.

(2) On the basis of the above parameters, we obtain

$$\begin{aligned} & \frac{\|Q\|T^{\gamma_2 - \gamma_1}}{\Gamma(\gamma_2 - \gamma_1 + 1)} + \frac{T^{\gamma_2 - 1/\rho_2^*} \|\theta_1\|_{L^{\rho_2^*}}}{\Gamma(\gamma_2)((\gamma_2 - 1)\rho_2 + 1)^{1/\rho_2}} + \\ & \frac{T^{\gamma_2 - 1/\rho_3^*} \|\theta_2\|_{L^{\rho_3^*}}}{\Gamma(\gamma_2)((\gamma_2 - 1)\rho_3 + 1)^{1/\rho_3}} \approx 0.2319 < 1. \end{aligned}$$

By Theorem 1, Problem (12) has a unique solution on $[-0.2, 1]$.

(3) It is easy to see that $\theta_1(t)$ and $\theta_2(t)$ are continuous and nondecreasing on $[0, 1]$. Based on the above data, the conditions of Theorem 3 are satisfied. Therefore, by Theorem 3, there exists a unique solution to Problem (12) on $[-0.2, 1]$.

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