

Mechanical, vibrational, and dynamical properties of amorphous systems near jamming

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Amorphous systems undergo the jamming transition when the density increases, temperature drops, or external shear stress decreases, as described by the jamming phase diagram which was proposed to unify different processes such as the glass transition, random close packing, and yielding under shear stress. At zero temperature and shear stress, the jamming transition occurs at a critical density at Point J . In this paper, we review recent studies of the material properties of marginally jammed solids and the glassy dynamics in the vicinity of Point J . As the only singular point in the jamming phase diagram, Point J exhibits special criticality in both mechanical and vibrational quantities. Dynamics approaching the glass transition in the vicinity of Point J show critical scalings, suggesting that the molecular glass transition and the colloidal glass transition are equivalent in the hard sphere limit. All these studies shed light on the long-standing puzzles of the glass transition and unusual properties of amorphous solids.

Keywords jamming, amorphous systems, critical phenomena, glass transition

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1 Introduction

We experience jamming in nature and our everyday life. For instance, traffic jams when too many vehicles run on the street; sand forms beautiful dunes in desert without sandstorm; honey freezes when kept in refrigerator. In condensed matter physics, jamming is a concept to describe the liquid–solid transition of amorphous materials

such as glasses, granular materials, colloidal suspensions, foams, and emulsions [1–5]. Although they are classified into different catalogs, these amorphous materials possess some common features. Firstly, they are disordered. Their microscopic structure is so random that no long-range order could be observed. During the transition from liquid-like to solid-like, the structure does not show any significant change. Secondly, they are viscoelastic. If an external force, e.g. a shear force, is applied, they initially respond elastically like a normal solid, but eventually flow like a liquid after the yield stress has been exceeded. Thirdly, they are out of thermal equilibrium. On the approach to the jamming transition from the liquid side, the relaxation time of the system grows up quickly with only a little decrease of the temperature or increase of the density, and eventually exceeds the time window that we can measure. In this sense, the jammed materials are non-equilibrium. All these unique properties of amorphous systems make them distinct from normal solids like crystals and lead to extreme difficulties in theoretic analysis. To understand the physics of amorphous systems is of great challenge in modern condensed matter physics.

In the framework of the jamming phase diagram proposed by Liu and Nagel to generalize the jamming of

various amorphous materials, jammed materials can be achieved by lowering the temperature T , increasing the density such as the volume fraction ϕ , or decreasing the applied external stress such as the shear stress σ_{xy} [6–8], as schemed in Fig. 1.

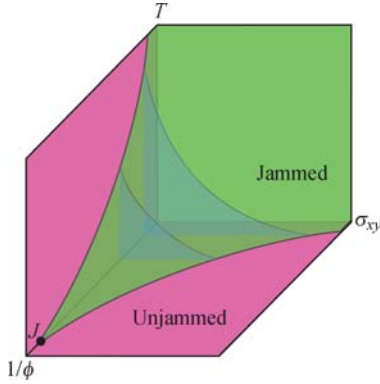


Fig. 1 Jamming phase diagram as a function of the temperature T , volume fraction ϕ , and shear stress σ_{xy} . Jammed region is marked as green. Point J is the jamming transition on the volume fraction axis.

In the T – ϕ plane, the jamming transition is accompanied with sluggish dynamics. If we try to avoid crystallization by quickly quenching the temperature of a liquid or increasing the volume fraction of a colloidal suspension, the viscosity and relaxation time of the liquid or colloidal suspension increase rapidly. A glass is formed when the relaxation time is longer than the measurable time window [9–12]. It is thus natural to expect a connection between the jamming transition and glass transition. Although a large number of studies have been carried out and some theories have been proposed as well [13–18], glass transition still remains elusive. On both sides of the glass transition, the glass-forming liquids and glasses exhibit unusual properties. For instance, glass-forming liquids show growing dynamical heterogeneity approaching the glass transition [19–22]; low-temperature glasses have anomalous vibrational properties, thermal conductivity and thermal capacity distinct from normal solids [23–27].

In the ϕ – σ_{xy} plane, jamming transition describes the yielding of jammed solids. When the applied shear stress is smaller than the yield stress, the jammed solid is able to sustain the stress and remains static, while it flows forever when the stress is above the yield stress. Due to the viscoelasticity, flow behaviors of amorphous systems are much more complicated than simple Newtonian fluids, which cannot be handled by kinetic theory. For instance, complex fluids, e.g. granular materials under shear, show shear banding [29–33] compared to the linear velocity profile in simple liquids; the relation between the shear stress and strain rate, i.e. the flow curve, is not linear as simple liquids do, which leads to the interesting shear-thinning and shear-thickening phenomena [34–36].

For systems with short-range interactions, there is a

critical point of the jamming transition at $T = 0$ and $\sigma_{xy} = 0$, denoted as Point J in the jamming phase diagram. It is the jamming transition of packings of frictionless spheres and, correspondingly, the close packing of disordered hard spheres. As implied by the jamming phase diagram, Point J is the unique singular point on the jamming transition surface, which may behave as a fix point and control jamming transition and properties of jammed systems at all temperatures and shear stresses. It is thus essential to understand the physics near Point J first.

The jamming phase diagram covers almost all regimes of the major research topics of amorphous systems: (i) solid-state properties of jammed solids, (ii) dynamics of glass-forming liquids, (iii) rheology of sheared systems, and (iv) statistical mechanics of aging and sheared systems. In this paper, we will focus on recent progress on study of properties of amorphous systems in the vicinity of Point J .

The structure of this paper is as follows. In Section 2, we will introduce a simple model to generate packings of frictionless spheres at $T = 0$ and $\sigma_{xy} = 0$. In Section 3, the distribution of the critical volume fraction at Point J and random close packing will be discussed. In Section 4, we will discuss the structural signatures in both the pair correlation function $g(r)$ and static structure factor $S(k)$ along with the jamming transition. Section 5 will focus on the critical scalings in marginally jammed solids near Point J . In Section 6, we will mainly concentrate on the vibrational properties of marginally jammed solids through the analysis of the normal modes of vibration. In Section 7, we will briefly discuss recent observation of the equivalence of the molecular glass transition and colloidal glass transition in the vicinity of Point J . Finally, conclusions will be drawn in Section 8.

2 Numerical model

The numerical system that we focus on is a d -dimensional box of length L consisting of N non-adhesive spheres. Periodic boundary conditions are applied in all directions to reduce the boundary effects. To avoid crystallization, a binary (50:50) mixture of particles with a diameter ratio $R > 1$ (bi-disperse) or a polydisperse mixture is usually employed. Three-dimensional mono-disperse ($R = 1$) systems can form jammed solids as well, as long as they are quenched fast enough to local energy minima to bypass crystallization. In two dimensions, mono-disperse systems are too easy to form crystals to be recommended as a typical model of jammed systems.

Particles interact via pairwise potentials. In order to guarantee a well-defined Point J , purely repulsive interactions are usually employed. Typical repulsions include harmonic ($\alpha = 2$) and Hertizan ($\alpha = 5/2$)

$$V(r_{ij}) = \frac{\epsilon}{\alpha} \left(1 - \frac{r_{ij}}{\sigma_{ij}} \right)^\alpha \quad (1)$$

or repulsive Lennard–Jones (Weeks–Chandler–Andersen) [37]

$$V(r_{ij}) = \frac{\epsilon}{72} \left[\left(\frac{\sigma_{ij}}{r_{ij}} \right)^{12} - 2 \left(\frac{\sigma_{ij}}{r_{ij}} \right)^6 + 1 \right] \quad (2)$$

where ϵ is the characteristic energy scale, r_{ij} is the separation between particles i and j , and $\sigma_{ij} = (\sigma_i + \sigma_j)/2$ is the sum of their radii. We set the small particle diameter $\sigma = 1$, particle mass $m = 1$, and $\epsilon = 1$, so the temperature is in the units of ϵ/k_B where k_B is the Boltzmann constant.

A more realistic Lennard–Jones inter-particle potential [38],

$$V(r_{ij}) = \frac{\epsilon_{ij}}{72} \left[\left(\frac{\sigma_{ij}}{r_{ij}} \right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij}} \right)^6 \right] \quad (3)$$

is also used in studies of glasses, where ϵ_{ij} is the interaction energy scale which depends on interacting species. Due to the attraction, there is no well-defined jamming transition of the Lennard–Jones model at $T = 0$ and $\sigma_{xy} = 0$.

Using this simple model, we are able to explore the whole jamming phase diagram. At $T = 0$ and $\sigma_{xy} = 0$, a packing of spheres can be obtained by fast quench of a random configuration to the local energy minimum using various energy minimizers such as conjugate gradient [37] and L-BFGS [40]. Figure 2 illustrates an unjammed and a jammed configuration of spheres obtained from the energy minimization. A jammed packing at $\Delta\phi = \phi - \phi_c$ is obtained via tuning the size of all particles simultaneously and repeating energy minimizations, where ϕ_c is the critical volume fraction at Point J . At $T > 0$ or $\sigma_{xy} > 0$, molecular dynamics simulations are required to solve the equations of motion numerically. Without the external shear stress, a thermostat or a barostat is necessary to maintain constant temperature or pressure [41]. When $\sigma_{xy} > 0$, a dissipation mechanism must be included in order to remove the energy pumped into the system continuously by the external shear force, so that a steady state is guaranteed. For instance, a damping

viscous force proportional to the relative velocities between interacting particles can be taken into account in equations of motion [32, 33, 42–44].

3 Random close packing of hard spheres and distribution of ϕ_c

For Point J , the first issue that is most likely to be addressed is the value of ϕ_c . In 1611, Kepler made a conjecture that the closest packing of equally sized spheres in three dimensions could not exceed the volume fraction of face-centered cubic close packing or hexagonal close packing, which is $\pi/\sqrt{18}$. Kepler’s conjecture was almost proved by Hales recently [45]. In 1960’s, Bernal observed experimentally that if equally sized hard spheres were randomly poured into a container, the volume fraction of jammed hard spheres was always around 0.64 [46] which was named as random close packing. In two dimensions, random close packing of bi-disperse hard disks with $R = 1.4$ is around 0.84 [47, 48]. The robustness of random close packing hints a possible existence of an elusive disordered version of the modern Kepler’s conjecture [49].

If particles are frictional, hard spheres jam at the so-called random loose packing, which is lower than random close packing and depends on the coefficient of restitution of the spheres [50, 51]. Friction has some interesting effects on jamming of spheres [50–53]. In this paper, we will only focus on systems without friction.

Although random close packing is robust for hard spheres, it has been debated whether random close packing is well-defined [54, 55]. It has been argued that hard spheres could jam at volume fractions ranging from < 0.64 to 0.74 , relying on the rate to generate the packing. However, the translational order of the packing varies with the volume fraction. Random close packing should thus be maximally disordered, so it was proposed to define packings of hard spheres at $\phi = 0.64$ as maximally random jammed states [54].

For packings of soft spheres, the jamming transition occurs at ϕ_c , which should be consistent with the random close packing of hard spheres. Recent studies have shown that less biased protocol, e.g. conjugate gradient energy minimization of random configurations at $T = \infty$, leads to a distribution of ϕ_c , $P_j(\phi_c)$ whose width decays with increasing the system size [48, 55], as shown in Fig. 3. In the $N \rightarrow \infty$ limit, $P_j(\phi)$ shrinks into a δ -function at random close packing, so almost all the possible jammed states exist at random close packing.

When N is finite, there are a finite number of jammed states at ϕ_c . Each jammed state occupies a basin with a volume in the configurational space [56]. The distribution of ϕ_c can thus be decomposed into the product of the distribution of distinct jammed states $\rho_j(\phi_c)$ and their

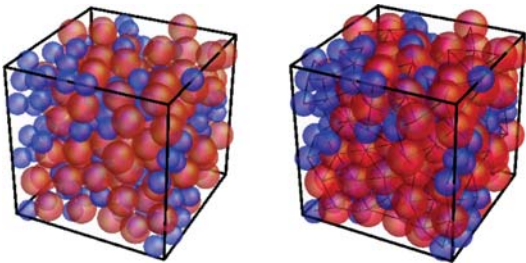


Fig. 2 Configuration of an unjammed (left) and a jammed (right) static packing of frictionless spheres with periodic boundary conditions. The particle size ratio is $R = 1.4$. The black lines in the right figure are bonds between interacting particles.

basins of attraction $\beta_j(\phi_c)$: $P_j(\phi_c) = \rho_j(\phi_c)\beta_j(\phi_c)$ [55]. $\rho_j(\phi_c)$ is intrinsic and does not vary with protocols, while $\beta_j(\phi_c)$ strongly depends on protocols. Recent studies indicate that $\rho_j(\phi_c)$ becomes narrower with increasing the system size and tends to approach a δ -function at the random close packing, similar to the system size dependence of $P_j(\phi_c)$ obtained from fast energy minimizations [48, 55]. It is thus reasonable to expect that random close packing is the most probable volume fraction for hard spheres to jam at. Hard spheres jam at volume fractions different from random close packing because $\beta_j(\phi_c)$ is biased by specific protocols.

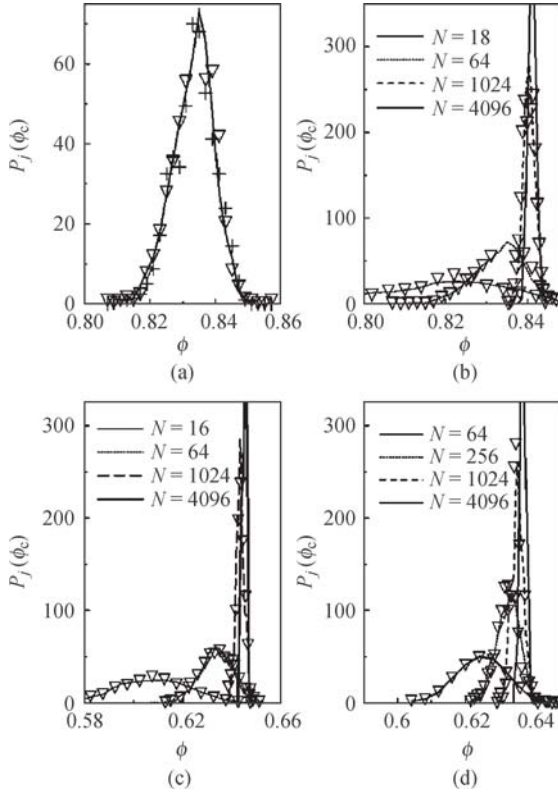


Fig. 3 (a) Distribution of jamming thresholds $P_j(\phi_c)$ for a 2-D bi-disperse system with $N = 64$ for the three different potentials studied [$\alpha = 3/2, 2$, and $5/2$, see Eq. (1)]. $P_j(\phi_c)$ for (b) 2-D bi-disperse systems, (c) 3-D bi-disperse systems, and (d) 3-D mono-disperse systems with harmonic and Hertzian potentials for various system sizes. In (a)–(d), the pluses, lines, and downward triangles represent potentials with $\alpha = 3/2, 2$, and $5/2$, respectively. Adapted from Refs. [47, 48], Copyright © 2002, 2003 the American Physical Society.

Random close packing also depends on the shape of particles and dispersion of particle sizes. For packings of elongated particles such as ellipsoids [57–60] and chains of spheres [61], ϕ_c is sensitive to the aspect ratio of the constituent elements. Moreover, for bi-disperse systems of spheres, it has been shown that when the particle diameter ratio R increases from 1, ϕ_c firstly increases up to a maximum at $R \sim 2$ and then decays [62]. The decay of ϕ_c is trivially due to the growing amount of rattlers (particles with no interacting neighbors). Although R affects ϕ_c , properties of jammed solids that will be discussed in

the following sections do not change qualitatively.

4 Structure of jammed solids

During the phase transition from a liquid to a crystal, apparent structural changes can be observed in both the pair correlation function

$$g(r) = \frac{L^d}{N^2} \left\langle \sum_i \sum_{j \neq i} \delta(\mathbf{r} - \mathbf{r}_{ij}) \right\rangle \quad (4)$$

where d is the dimension of space, and $\langle \dots \rangle$ is the ensemble average, and static structure factor

$$S(k) = \frac{1}{N} \left| \sum_j \exp(i\mathbf{k} \cdot \mathbf{r}_j) \right|^2 \quad (5)$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ is the wave vector, due to the break of symmetry. For instance, $g(r)$ and $S(k)$ are spiky in solid state, while continuous in liquid state. On the contrary, there are no significant structural changes during the jamming transition at Point J . $g(r)$ and $S(k)$ look similar between unjammed and jammed states. However, there are still some observable features in both $g(r)$ and $S(k)$ that may signify the jamming transition.

The first feature is that the second peak of $g(r)$ usually present in a liquid at $r = 2$ splits into subpeaks. The number of subpeaks and their positions depend on the particle size dispersion [62]. As shown in Fig. 4, for mono-disperse systems ($R = 1$), the subpeaks occur at $r = \sqrt{3}$ and 2, which correspond to the specific structures illustrated in Fig. 4(b) and (c). It is still puzzling why particles tend to form these clusters approaching jamming. For bi-disperse ($R > 1$) systems, there are more subpeaks due to the more complicated conformations formed by particles with different sizes. The split of the second peak happens when $\phi < \phi_c$ and in glass-forming liquids as well, which has been proposed as the signal of the glass transition, but the right-hand side of the subpeaks becomes steeper when Point J is approached from either side. At Point J , the right-hand side of the subpeaks drops discontinuously because particles are just in touch with each other at the jamming transition.

The first peak of $g(r)$ at $r = 1$ shows similar singularity at Point J . On the approach to Point J from both sides, the height of the first peak, g_1 , increases, while the width of the peak shrinks. At Point J , g_1 diverges, and the first peak of $g(r)$ becomes a δ -function. The discontinuity of the first peak and the subpeaks apparently signifies the jamming transition at Point J .

At $T > 0$, g_1 exhibits a maximum when the volume fraction (pressure) is varied [8, 63]. As shown in Fig. 5, the maximum of g_1 , $g_1^{\max}$ increases with decreasing the temperature and diverges at Point J when $T = 0$. Therefore, the presence of $g_1^{\max}$ in thermal systems rem-

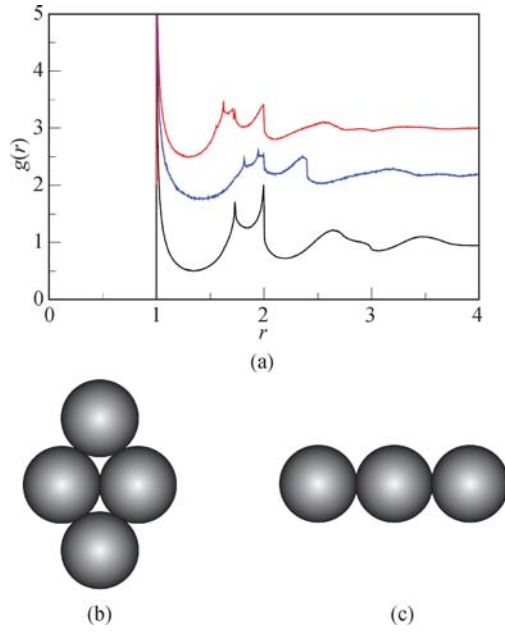


Fig. 4 (a) Pair correlation function $g(r)$ for jammed systems with $R=1$ (bottom, black) and $R = 1.4$ at pressure $p = 10^{-4}$. For $R = 1.4$, $g(r)$ is measured for both large particles (top, red) and small particles (middle, blue). r is scaled with the (small) particle diameter. $g(r)$ for $R = 1.4$ systems are shifted up in order to distinguish the curves. (b) and (c) are conformations of particles that form the subpeaks at $r = \sqrt{3}$ and 2 for $R = 1$ systems. Adapted from Ref. [62], Copyright © 2010 the Royal Society of Chemistry.

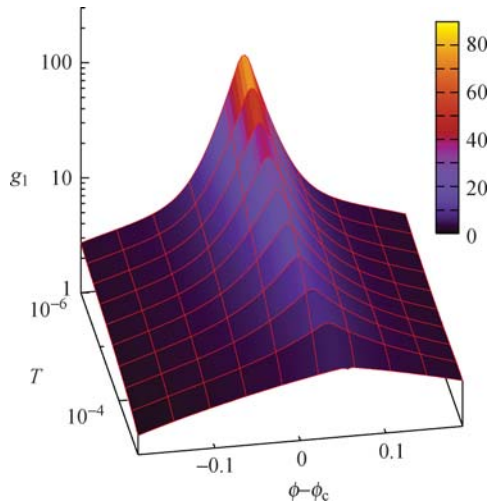


Fig. 5 Schematic plot of the height of the first peak in $g(r)$, g_1 versus temperature T and volume fraction $\phi - \phi_c$ for bi-disperse systems ($R = 1.4$) with harmonic repulsion. Adapted from Ref. [8], Copyright © 2009 the Nature Publishing Group.

inises the $T = 0$ jamming transition, at which material properties, e.g. pressure (or bulk modulus) and shear modulus, show crossover between two distinct regimes with different volume fraction dependence and, hence, signify a transition [63]. Remarkably, $g_1^{\max}$ emerges only when the volume fraction (pressure) changes. At fixed volume fraction (pressure), g_1 increases monotonically with decreasing the temperature. Note that the presence of $g_1^{\max}$ is not coincident with the glass transition

[8, 63], so it is still unclear if the glass transition and jamming transition are the same processes as initially expected.

In the long wave-length limit ($k \rightarrow 0$), the static structure factor $S(k)$ of jammed solids shows distinct k dependence from liquids. In liquid state, $S(k)$ is quadratic in k , and $S(0)$ which is proportional to the isothermal compressibility in thermal equilibrium systems is usually greater than zero. However, in mono-disperse jammed systems, it has been observed that $S(k) \sim k$ in the long wave-length limit and $S(0) = 0$, implying that jammed packings of frictionless spheres are hyperuniform [64, 65]. Recent studies indicate that the long wave-length behavior of $S(k)$ relies on the particle size ratio R in bi-disperse jammed systems [62]. The linear $S(k)$ is the limiting behavior at $R = 1$. In jammed solids, $S(k)$ does not show observable $\Delta\phi$ dependence in the long wave-length limit. It is thus puzzling if there exists a static correlation length associated with the jamming transition at Point J [66].

5 Criticality of point J

The jamming transition at Point J separates liquid-like unjammed systems from jammed solids. When $\phi < \phi_c$, particles do not touch and are free to move without consuming any energy, so the unjammed systems at $T = 0$ have zero bulk modulus and shear modulus. When $\phi > \phi_c$, particles have to overlap and deform. A force network percolates through the system. If we attempt to move one network particle, the potential energy of the system must rise up to overcome the constraints. A jammed system behaves like a solid with nonzero bulk modulus and shear modulus. This liquid-solid transition implies the criticality of Point J . In the following, we will see critical scalings in marginally jammed solids near Point J . However, the criticality at Point J is unusual and, hence, could not be fully understood from conventional phase transition theories.

Assuming the simple power-law repulsion [Eq. (1)] between interacting particles, in marginally jammed solids, the pressure p , bulk modulus $B = \phi dp/d\phi$, static shear modulus G_∞ measured in the infinite waiting time limit, and coordination number (average interacting neighbors per particle) Z are all scaled well with $\Delta\phi = \phi - \phi_c$ [47, 48, 67, 68],

$$p \sim (\Delta\phi)^{\alpha-1} \quad (6)$$

$$B \sim (\Delta\phi)^{\alpha-2} \quad (7)$$

$$G \sim (\Delta\phi)^{\alpha-3/2} \quad (8)$$

$$Z - Z_c \sim (\Delta\phi)^{1/2} \quad (9)$$

where $Z_c = 2d$ is the isostatic value of Z with d the dimension of space. Figure 6 shows p , G_∞ , and $Z - Z_c$ for

both harmonic ($\alpha = 2$) and Hertzian ($\alpha = 5/2$) systems in both 2-D and 3-D. Eqs. (6)–(9) are all well satisfied. These critical scalings imply the criticality of Point J . However, the scalings show some anomalies that distinguish the jamming transition from conventional phase transitions. Firstly, some of the scalings are a function of inter-particle potential. Secondly, the ratio of the moduli $G/B \approx (\Delta\phi)^{1/2}$ varies with $\Delta\phi$ and decays to zero at Point J . Thirdly, the jamming transition at Point J mixes up first- and second-order phase transitions. For harmonic repulsion ($\alpha = 2$), Eq. (7) indicates that the bulk modulus jumps discontinuously from 0 to a nonzero value at Point J , implying a first-order transition, so does the coordination number. However, the shear modulus

still increases continuously from 0 starting at Point J [see Eq. (8)], implying a second-order transition.

Isostaticity, i.e. $Z = Z_c$, is one of the most extraordinary properties of Point J , which controls vibrational properties of marginally jammed solids and will be discussed in the next section. Consider a static system consisting of N frictionless spheres in d dimensions. The total number of independent variables in the system is Nd . In order to freeze all these degrees of freedom, we need at least Nd constraints, while there are totally $NZ/2$ inter-particle bonds. The minimum coordination number must be $Z = 2d$ ($Nd = NZ/2$) to maintain mechanical stability of the whole system, which is exactly the isostatic value. Therefore, jammed systems at Point J have barely enough number of constraints to satisfy the minimum requirement of mechanical stability.

The criticality of Point J can be seen as well from the scaling functions observed in shear flows [69, 70]. Molecular dynamics simulations have shown that the viscosity η of a shear flow with harmonic repulsion can be fitted with a scaling function of the shear stress σ_{xy} [69],

$$\eta^{-1}(\phi, \sigma_{xy}) = |\phi - \phi_c|^\beta f_{\pm} \left(\frac{\sigma_{xy}}{|\phi - \phi_c|^\Delta} \right) \quad (10)$$

where f_+ and f_- are scaling functions for jammed ($\phi > \phi_c$) and unjammed ($\phi < \phi_c$) systems. The scaling exponents are approximately $\beta = 1.7 \pm 0.2$ and $\Delta = 1.2 \pm 0.2$. Correspondingly, there is a diverging length at Point J [69],

$$\xi^{-1}(\phi, \sigma_{xy}) = |\phi - \phi_c|^\nu h_{\pm} \left(\frac{\sigma_{xy}}{|\phi - \phi_c|^\Delta} \right) \quad (11)$$

where h_+ and h_- are scaling functions for jammed and unjammed systems, and the scaling exponent $\nu = 0.6 \pm 0.1$.

In shear flow at Point J , the shear stress and pressure also satisfy the finite size scalings [70],

$$\sigma_{xy}(\gamma, L) = L^{-1/y_\sigma} f_\sigma \left(\gamma L^{1/y_\gamma} \right) \quad (12)$$

$$P(\gamma, L) = L^{-1/y_P} f_P \left(\gamma L^{1/y_\gamma} \right) \quad (13)$$

where γ is the shear rate, and the scaling exponents $1/y_\sigma = 2.5 \pm 0.2$, $1/y_P = 2.0 \pm 0.2$, and $1/y_\gamma = 4.0 \pm 0.5$. A correlation length ξ_L exists and diverges in the $\gamma \rightarrow 0$ limit [70],

$$\xi_L \sim \gamma^{-y_\gamma} \quad (14)$$

A length scale that diverges at Point J has been reported in a number of simulations and experiments approaching Point J from either side [71–79]. Figure 7 shows the regime in jammed solids responding to a point force acting at the center of the system [75]. Apparently, the size of the observable response is larger close to Point J , which provides us with a direct illustration

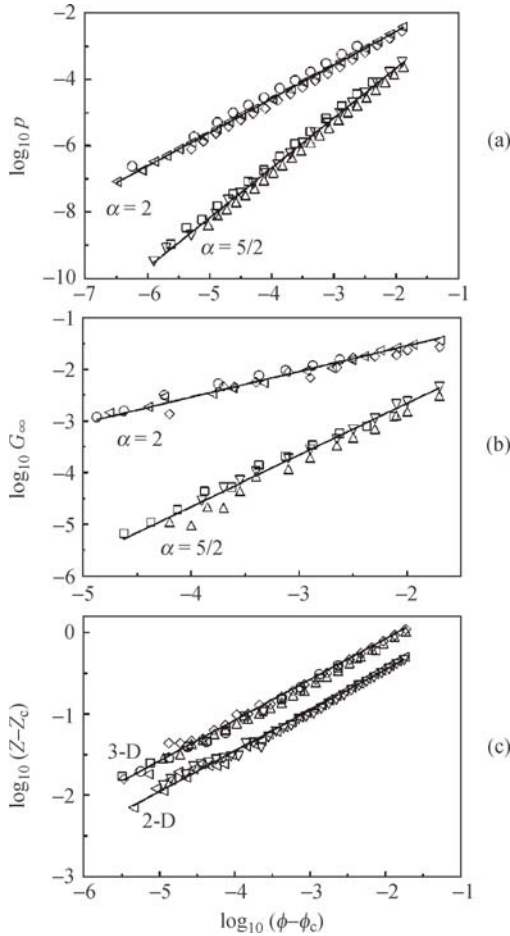


Fig. 6 (a) Pressure p , (b) static shear modulus G_∞ versus $\phi - \phi_c$ for 3-D mono-disperse (circles), 3-D bi-disperse (diamonds, $R = 1.4$), and 2-D bi-disperse (leftward triangles, $R = 1.4$) systems with harmonic repulsion ($\alpha = 2$, upper curves) and Hertzian repulsion ($\alpha = 5/2$, lower curves). (c) shows the extra coordination number over isostatic value $Z - Z_c$ versus $\phi - \phi_c$ for 3-D mono-disperse harmonic (circles), 3-D mono-disperse Hertzian (squares), 3-D bi-disperse harmonic (diamonds), 3-D bi-disperse Hertzian (upward triangles), 2-D bi-disperse harmonic (leftward triangles), and 2-D bi-disperse Hertzian (rightward triangles) systems. $N = 1024$ (512) particles were used for the 2-D (3-D) systems. The lines have a slope of 1.0 (upper) and 1.5 (lower) in (a), 0.5 (upper) and 1.0 (lower) in (b), and 0.5 in (c). Adapted from Refs. [47, 48], Copyright © 2002, 2003 the American Physical Society.

about the growing length approaching Point J . However, up to now, all the observed length scales are dynamical. As discussed in Section 4, the structure of jammed systems does not differ much from unjammed systems. There has been no report on the observation of a static correlation length diverging at Point J . Density correlation and susceptibility such as the response of particle moves to external perturbations do not show apparent volume fraction dependence [62, 65, 66]. It is thus mysterious whether there exists a diverging correlation length at Point J and what the correct order parameter is to present such a length, which distinguishes the jamming transition at Point J from conventional phase transitions.

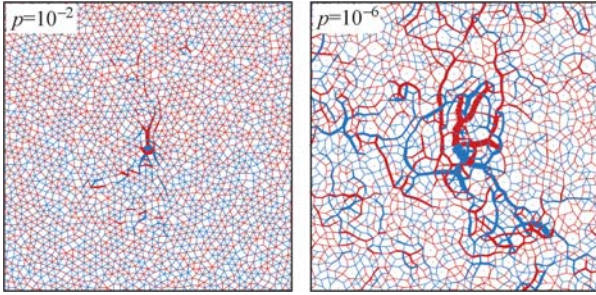


Fig. 7 Force response networks for a point loading with pressure as indicated. Blue (red) lines indicate positive (negative) changes in contact force, the thickness indicating the amount. The particles themselves are not drawn. The systems are bi-disperse at $R = 1.4$. Adapted from Ref. [75], Copyright © 2006 the American Physical Society.

6 Vibrations in jammed solids

In normal solids like crystals, particle vibration excites plane-wave phonons carrying and transporting energy. To study properties of a solid, it is fundamental to understand the vibrational properties through the measure of normal modes of vibration. In crystals, the density of states $D(\omega)$, i.e. number of normal modes as a function of frequency ω , obeys the Debye law: $D(\omega) \sim \omega^{d-1}$, where d is the dimension of space. In the absence of impurities or defects, normal modes in crystals are propagating phonons. Modes at high frequencies become localized (Anderson localization [80]) when impurities or defects exist.

In amorphous solids such as glasses, vibrations are much more complicated because of the disordered nature of the structure. Phonons are scattered due to the loss of periodicity. One of the most striking features of vibrations in some glasses and other amorphous solids is the boson peak in the low-frequency density of states and low-temperature heat capacity [23, 24, 27]. As shown in Fig. 8, density of states scaled by the Debye law exhibits a broad peak at low frequencies, implying the existence of excess vibrational modes beyond Debye prediction. The origin of the boson peak has been intensively studied

[81–90], but still remains elusive.

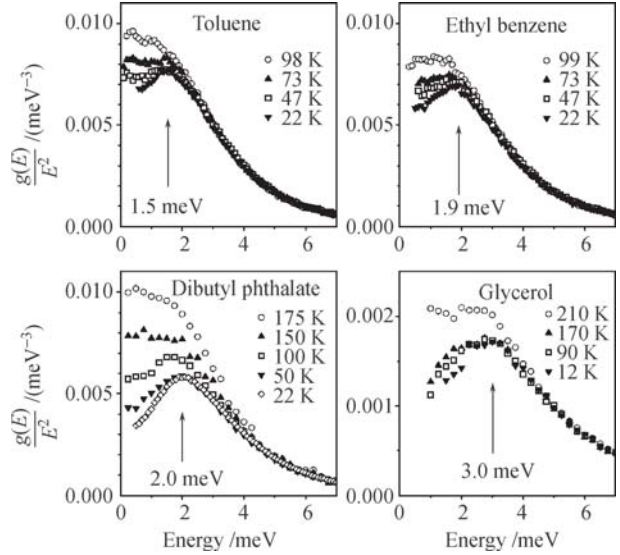


Fig. 8 Reduced density of states, $g(E)/E^2$ (note that we use $D(\omega)$ to denote the density of states) of collective motions in toluene, ethylbenzene, dibutylphthalate, and glycerol glasses, where E is the energy. Arrows indicate the energy of the boson peak estimated from the data at lowest temperature. Adapted from Ref. [27], Copyright © 2004 the American Physical Society.

Another anomalous feature of glasses is the unusual low-temperature thermal properties. As shown in Fig. 9 of the thermal conductivity of glasses, unlike the T^3

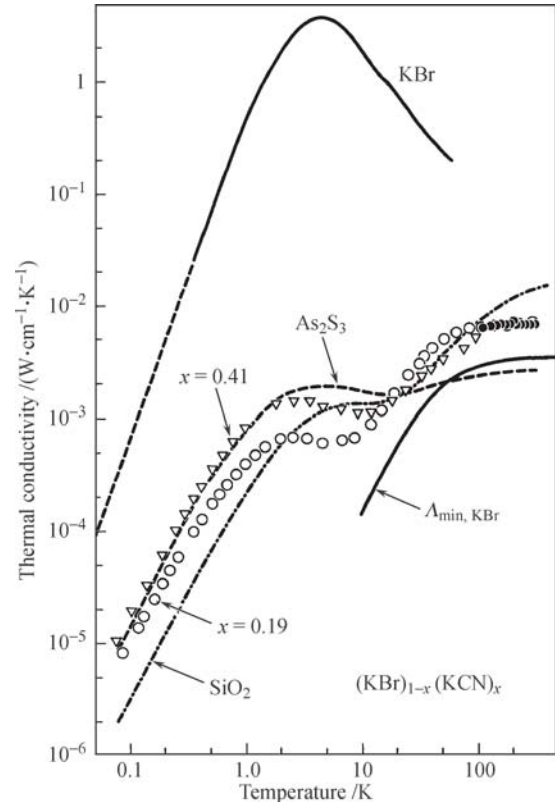


Fig. 9 Thermal conductivity of $\text{KBr}_{1-x}\text{KCN}_x$ for $x = 0$ (pure KBr), 0.19, and 0.41 compared to data for glasses. Note the differences between crystal ($x = 0$) and glasses ($x = 0.19$ and 0.41, As_2S_3 and SiO_2). Adapted from Ref. [28], Copyright © 1992 the American Physical Society.

behavior presented in crystals, the thermal conductivity in glasses is T^2 at low temperatures, followed by a plateau and a rise afterwards. The thermal conductivity of glasses increases with the temperature monotonically and saturates to a constant value at ordinary temperatures, in contrast to the approximately $1/T$ decay in crystals. Quantum two-level systems, i.e. neighboring anharmonic local energy minima, have been proposed to understand the T^2 scaling of the thermal conductivity at low temperatures [23, 91]. It has also been proposed that the plateau in the thermal conductivity of glasses arises from the boson peak. The increase of the thermal conductivity at high temperatures is expected if there exists a frequency-independent phonon mean free path or, equivalently, a constant energy diffusivity [92]. Therefore, it is a key to study the boson peak and energy transport in glasses in order to understand the unusual thermal properties of glasses.

As a simple model glass, marginally jammed solids possess most of the characteristics of realistic glasses. In this section, we discuss the nature of the normal modes in marginally jammed solids. We will see that it is the isostaticity that determines the properties of normal modes in marginally jammed solids, and, hence, probably the boson peak and unusual thermal properties of realistic glasses.

6.1 Excess normal modes and boson peak

The normal modes of vibration are obtained from the diagonalization of the dynamical matrix H . The entry of H is denoted as $H_{ia,jb}$, where i and j refer to particles i and j , and $a, b = x, y, z$ represent the Cartesian coordinates. When $i \neq j$,

$$H_{ia,jb} = -\frac{V'(r_{ij})}{r_{ij}}(\delta_{ab} - \hat{r}_{ija}\hat{r}_{ijb}) - V''(r_{ij})\hat{r}_{ija}\hat{r}_{ijb} \quad (15)$$

while

$$H_{ia,ib} = -\sum_j M_{ia,jb} \quad (16)$$

when $i = j$, where V' and V'' are the first and second derivatives of the potential $V(r)$ with respect to r , and $\mathbf{r}_{ij} = r_{ij}\hat{\mathbf{r}}_{ij}$ is the equilibrium separation vector between particles i and j . The eigenvalues and eigenvectors of the dynamical matrix are the energy of the modes ω_l^2 (note that we have set the mass of particles $m = 1$) and polarization vector $\mathbf{e}_l$, i.e.

$$H\mathbf{e}_l = \omega_l^2 \mathbf{e}_l \quad (17)$$

where ω_l is the vibrational frequency of mode l . For a system of N particles in d dimensions, there are totally dN modes. d of them are zero-frequency modes (soft modes) due to the translational invariance of the periodic boundary conditions. If there are rattlers, i.e.

particles without any constraints, each rattler induces d soft modes.

The density of states $D(\omega)$ of a jammed solid exhibits three typical regimes, as described by Fig. 10 of the density of states of jammed solids with harmonic repulsion [71]. At low frequencies, $D(\omega)$ rises quickly from zero. A plateau in $D(\omega)$ is developed after the low-frequency rise, which is followed by a fast decay to zero due to the Anderson localization. The plateau in $D(\omega)$ implies that excess normal modes (anomalous modes) beyond the Debye prediction emerge at low-frequencies and form the boson peak. An interesting observation is that the plateau extends to a lower frequency when $\Delta\phi$ decreases. The crossover frequency between the low-frequency rise and the plateau in $D(\omega)$, ω^* , is scaled well with the compression $\Delta\phi$ [71–73],

$$\omega^* \sim (\Delta\phi)^{1/2} \quad (18)$$

for jammed solids with harmonic repulsion. It is surprising that at Point J , where zero-temperature jamming transition happens, the density of states does not vanish at $\omega = 0$, so the system is on the verge of instability.

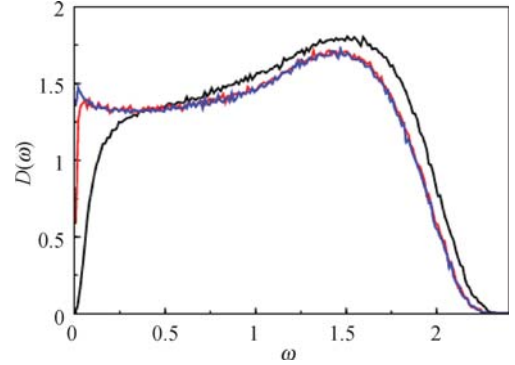


Fig. 10 Density of states of marginally jammed bi-disperse ($R = 1.4$) solids consisting of 1000 particles with harmonic repulsion at $\Delta\phi = 10^{-2}$ (black), 10^{-4} (red), and 10^{-6} (blue).

At Point J , the system is isostatic, but still marginally rigid. To remove any constraint in the isostatic system would induce soft modes, i.e. normal modes along which the system could not resist external forces. Consider a slightly compressed marginally jammed solid in 3-D with an extra coordination number beyond isostaticity, $\Delta Z = Z - Z_c$. The total number of removable bonds before losing rigidity is $N_r = N\Delta Z/2 \sim L^3\Delta Z$. If we cut all the bonds in a plane across the whole system, there are about $N_c \sim L^2$ bonds that are cut. There is a critical value of L , l^* , at which $N_r = N_c$: $l^* \sim 1/\Delta Z$. l^* is thus the wavelength of the anomalous modes that form the plateau in $D(\omega)$. The crossover frequency

$$\omega^* \sim v_s/l^* \sim v_s\Delta Z \quad (19)$$

where v_s is the speed of sound [72, 73, 93]. v_s is proportional to the square root of either the bulk modulus B or the shear modulus G , depending on the transverse or

longitudinal character of the wave. Assuming $v_s \sim \sqrt{B}$, from Eqs. (7), (9) and (19), we obtain

$$\omega^* \sim (\Delta\phi)^{(\alpha-1)/2} \quad (20)$$

which agrees with Eq. (18) observed in systems with harmonic repulsion ($\alpha = 2$). Equation (20) may imply that longitudinal character is dominant in anomalous modes.

In a jammed solid, any particle displacement $\delta\mathbf{R}_i$ from equilibrium positions would induce an energy rise [72, 73, 93]

$$\delta E = \frac{1}{2} \sum_{i,j} \left[V''(r_{ij})(\delta\mathbf{R}_{ij} \cdot \hat{\mathbf{r}}_{ij})^2 + \frac{V'(r_{ij})}{r_{ij}} (\delta\mathbf{R}_{ij}^\perp)^2 \right] \quad (21)$$

where $\delta\mathbf{R}_{ij} = \delta\mathbf{R}_i - \delta\mathbf{R}_j$, and $\delta\mathbf{R}_{ij}^\perp$ is the projection of $\delta\mathbf{R}_{ij}$ on the plane perpendicular to $\hat{\mathbf{r}}_{ij}$. The first term on the right-hand side of Eq. (21) arises from the head-to-head compressive motion of the interacting particles, while the second term corresponds to the slide between them. In order to simplify the analysis, we sometimes remove the force between interacting particles i and j and replace the interaction with an unstretched spring with a spring constant equal to $V''(r_{ij})$. This is equivalent to setting $V' = 0$, so that the second term in Eq. (21) vanishes. The system thus has zero potential energy, but is still marginally rigid. We name such a simplified system as *unstressed* system, while the original one is called *stressed* system [71–73, 78, 79, 94, 95].

For a system with a coordination number Z , we divide all the $NZ/2$ particle interactions into two parts: $NZ_1/2$ stiffest ones and $N(Z - Z_1)/2$ weak ones. Equation (21) can thus be approximated into [93, 94]

$$\delta E = A_1 k_1 (Z_1 - Z_c)^2 + \frac{1}{Nd} \sum_{ij}^w V''(r_{ij}) + \frac{d-1}{dN} \sum_{ij} \frac{V'(r_{ij})}{r_{ij}} \quad (22)$$

for anomalous modes in the vicinity of ω^* , where k_1 is the average stiffness ($\langle V'' \rangle$) over the $NZ_1/2$ stiffest interactions, $\sum_{ij}^w$ is over the $N(Z - Z_1)/2$ weak ones, and A_1 and A_3 are constants to be determined by the energy variation method. In *unstressed* systems, the third term on the right-hand side of Eq. (22) is zero.

In marginally jammed solids, because Z is slightly greater than Z_c and the stress (V') is negligibly small, only the first term on the right-hand side of Eq. (22) contributes to the vibrational energy of the anomalous modes substantially. δE is equal to $(\omega^*)^2$, so that $\omega^* \sim \sqrt{k_1}(Z - Z_c)$. k_1 is a function of α . Take a marginally jammed solid with harmonic repulsion ($\alpha = 2$) as an example. Because k_1 is constant in $\Delta\phi$, $\omega^* \sim (Z - Z_c)$, in agreement with Eqs. (9) and (20).

In realistic glasses in which the coordination number is

much greater than Z_c due to the long-range attractions or high densities, Eq. (22) still works well to predict the location of the excess vibrational modes, i.e. boson peak frequency ω^* . In Fig. 11, we compare ω^* calculated from Eq. (22) with ω^+ extracted directly from $D(\omega)$ for *unstressed* repulsive Lennard–Jones [Eq. (2)] and Lennard–Jones [Eq. (3)] glasses. They match very well for both types of realistic model glasses. Note that the number of stiffest interactions that determine the structure of the anomalous modes, Z_1 , is only slightly larger than the isostatic value Z_c , even though the coordination number of these realistic glasses is large. Therefore, most of the particle interactions just act as perturbations. The emergence of the boson peak in amorphous solids is closely related to the isostaticity at Point J , which has also been verified in lattices with randomness [96, 97]. It is thus a robust evidence to support the argument that the jamming transition at Point J controls the properties of glasses in the jammed regime on the T - ϕ plane of the jamming phase diagram (Fig. 1).

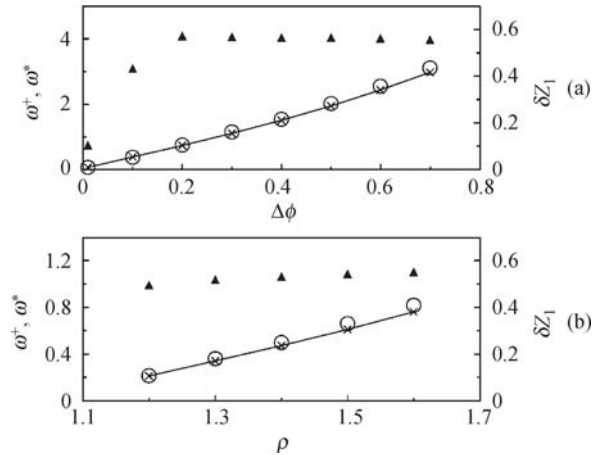


Fig. 11 Left axis: The characteristic frequency ω^+ (open symbols) calculated from numerical simulations and the corresponding theoretical predictions ω^* (crosses). Lines are to guide the eye and connect the theoretical points. Right axis: the fractional deviation from isostaticity $\delta Z_1 = (Z_1 - Z_c)/Z_c$ (closed symbols), as functions of (a) distance from the jamming transition, $\Delta\phi$, for repulsive Lennard–Jones mixtures, and (b) density ρ for Lennard–Jones mixtures. Adapted from Ref. [94], Copyright © 2007 the American Physical Society.

The presence of friction lowers the requirement of isostaticity to form marginally rigid systems, so that frictional systems can jam at $Z_c^\mu < Z_c$, where Z_c^μ is the minimally allowed coordination number to jam with the coefficient of friction μ . However, the critical scaling of the boson peak frequency ω^* with respect to the extra coordination number $Z - Z_c^\mu$ is still the same as discussed above [52].

Systems consisting of ellipsoids can jam at a coordination number lower than the isostaticity as well when the ellipticity is small. The vibrational spectrum is thus more complicated due to the presence of the rotational degrees of freedom. Two bands separated by a frequency

gap are formed in the density of states of jammed ellipsoidal systems [59, 60]. The rotational modes aggregate at low frequencies, while the translational modes form a density of states which is similar to jammed systems of spheres with an onset frequency of ω^* . ω^* is still scaled well with the extra coordination number with respect to $Z_c = 2d$, implying that the jammed systems of spheres control the vibrational properties of the ellipsoidal systems in an unusual and nontrivial way.

6.2 Quasi-localization and anharmonicity

From the density of states, we are able to predict that several different types of normal modes exist in jammed solids. As mentioned above, $D(\omega)$ decays to zero at the high-frequency end of the spectrum, reminding us of the Anderson localization due to the presence of disorder. The boson peak frequency ω^* seems to separate anomalous modes at $\omega > \omega^*$ which form the plateau in $D(\omega)$ and a type of low-frequency modes at $\omega < \omega^*$.

The extensiveness of a mode affects its ability to conduct energy, which can be quantified by the participation ratio, i.e. fraction of particles involved in the vibration when the mode is excited. The participation ratio of mode l is defined as:

$$p_l = \frac{(\sum_i e_{l,i}^2)^2}{N \sum_i e_{l,i}^4} \quad (23)$$

where the sum is over all particles. For a plane-wave phonon, the participation ratio is 1, while a localized mode has a vanishing participation ratio.

Figure 12 shows the participation ratio versus frequency of a 3-D marginally jammed solid with harmonic repulsion [95, 98, 99]. There are apparently three distinct regimes in the *unstressed* case: (i) localized modes at high frequencies with a vanishing participation ratio, (ii) extended anomalous modes in the intermediate frequency regime with a roughly frequency-independent participation ratio (~ 0.5), and (iii) plane-wave like modes at low frequencies with the participation ratio increasing toward zero frequency. Regimes (i) and (ii) persist in the stressed case, but regime (iii) is not so

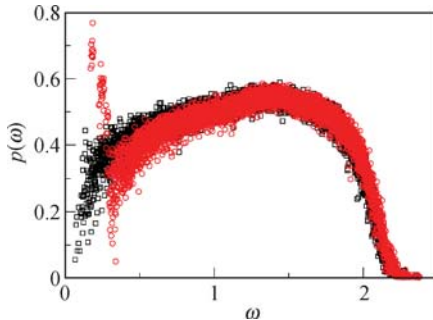


Fig. 12 Participation ratio $p(\omega)$ of a 3-D jammed bi-disperse ($R = 1.4$) solid at $N = 2000$ and $\Delta\phi = 0.1$ with harmonic repulsion. The red circles and black squares are for unstressed and stressed cases, respectively.

obvious due to the finite size effect.

A very special feature that we can see from Fig. 12 is the dip at the crossover between regimes (ii) and (iii). Modes at this crossover are quasi-localized with a small but non-vanishing participation ratio. Compared to the density of states, we can tell that quasi-localized modes aggregate in the vicinity of the boson peak frequency ω^* . The quasi-localized modes persist approaching the unjamming transition. They move to lower frequencies with decreasing the volume fraction, the same way as ω^* does. Quasi-localized modes have also been observed in other model glasses [100, 101]. It has been shown that the quasi-localized modes at the boson peak may control the properties of glasses and the glass transition [95, 102], so it is of great importance to discover the origin of the quasi-localization.

Figure 13 visualizes the three types of modes in a two-dimensional jammed solid with harmonic repulsion. The modes in regime (iii) look like weakly scattered phonons. Plane-wave-like structure hybridizes with some localized regions, which leads to distortions of the plane-wave structure and hence quasi-localization. The structure of the modes in regime (ii) is random. There will be no correlated motion when these modes are excited. A perturbation, e.g. a sound wave, will be strongly scattered and dissipated via these modes. In localized modes of regime (i), only several particles are substantially involved in vibrations. These modes thus cannot conduct any energy at all.

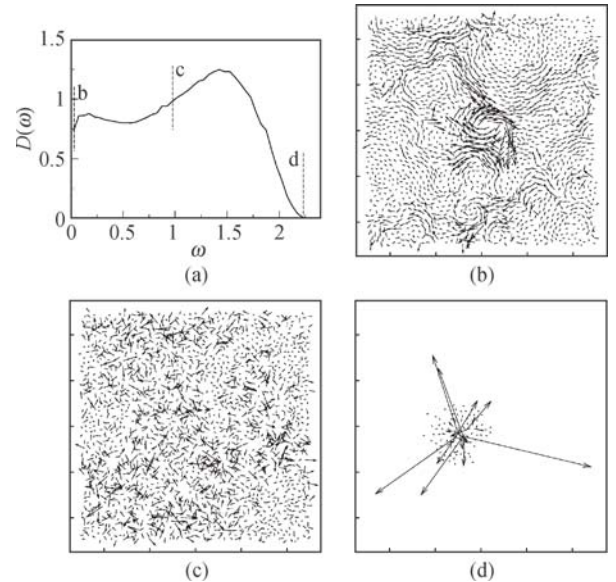


Fig. 13 (a) Density of states $D(\omega)$ of a 2-D jammed bi-disperse ($R = 1.4$) solid at $\Delta\phi = 10^{-4}$ and $N = 2000$ with harmonic repulsion. (b), (c), and (d) are pictures of the normal modes of vibration at frequencies labeled in (a) by the vertical dashed lines.

Besides the quasi-localization, the modes in the vicinity of ω^* behave strong anharmonicity [95]. For instance, the Grüneisen parameter of these modes, $\gamma_g = \frac{d \ln \omega}{d \ln \phi}$,

shows drastic frequency dependence, while in crystals with harmonic repulsion, γ_g is almost constant and small in frequency. Moreover, the energy barrier along these modes before falling into a neighbor local energy minima is the lowest. Therefore, the quasi-localized modes near ω^* are the easiest to cause instability. Under thermal perturbations, a jammed solid melts easily along these modes, so understanding the anharmonicity of the quasi-localized modes is important to discover the origin of the glass transition in the framework of the potential energy landscape. Furthermore, the strong anharmonicity of the quasi-localized modes may also have some implications to the two-level systems proposed to explain the low-temperature thermal properties of glasses [91], which requires further investigations.

6.3 Energy transport

Since the nature of the normal modes in jammed solids strongly depends on frequency, modes in different frequency regimes will have different ability to transport energy. For example, when a sound wave is injected into a jammed solid, e.g. granular materials, the outgoing signal shows strong frequency dependence [103–106]. In a normal solid, the thermal conductivity is approximately $\kappa = \frac{1}{3}cv_sl = cd$, where c is the heat capacity, v_s is the speed of sound, l is the phonon mean free path, and $d = v_sl/3$ is the energy diffusivity. The energy diffusivity is thus a fundamental quantity to measure in order to evaluate the energy transport in amorphous solids.

Consider a wavepacket that is narrowly peaked at a frequency ω initially and spreads out in time. The diffusivity $d(\omega)$ measures how fast the width of the wavepacket is broadened. It can be calculated from the Kubo-Greenwood formula as long as the normal modes are known [107, 108].

Similar to what have been observed in the density of states and participation ratio of marginally jammed solids, there are correspondingly three frequency regimes of $d(\omega)$. As shown in Fig. 14, the high-frequency localized modes have a vanishing diffusivity, so the localized modes cannot transport any energy; in the intermediate frequency regime above a crossover frequency ω_d , the diffusivity is small but almost constant in frequency; below ω_d , the diffusivity diverges toward $\omega = 0$ approximately according to the Rayleigh scattering, i.e. $d(\omega) \sim \omega^{-4}$, indicating that these modes are weakly scattered plane waves. In stressed cases, the Rayleigh scattering is hardly seen due to the finite size effect.

The flat diffusivity above ω_d is of special interest in our understanding of the monotonic increase of the thermal conductivity of glasses near room temperature, as what Kittel predicted [92]. Figure 14 shows that ω_d decreases with decreasing $\Delta\phi$ and approaches zero at Point J . It

is thus another robust evidence that the properties of glasses may be understood from the underlying physics near the jamming transition at Point J .

The crossover frequency ω_d between the Rayleigh scattering and flat diffusivity in the diffusivity spectrum can be understood from the Ioffe-Regel limit [109], i.e. crossover between weakly and strongly scattered transverse modes. Define ω_{IR} as the Ioffe-Regel frequency that can be predicted from the Ioffe-Regel limit. At the Ioffe-Regel crossover, $kl \sim 1$. Constant diffusivity d_0 means that $v_sl/3 = d_0$ at ω_{IR} . The combination of these two conditions gives $\omega_{IR} \sim v_s^2/(3d_0)$. Dimensional analysis shows that $d_0 \sim \sigma^2 \sqrt{V''}/m \sim (\Delta\phi)^{(\alpha-2)/2}$ [78, 79]. For the transverse modes, the speed of sound v_s is proportional to $(\Delta\phi)^{\alpha/2-3/4}$ [see Eq. (8)]. Therefore, $\omega_{IR} \sim (\Delta\phi)^{(\alpha-1)/2}$, in excellent agreement with the scaling of ω^* shown in Eq. (20). As confirmed by the inset to Fig. 14(a), the boson peak frequency ω^* , the crossover frequency estimated directly from the diffusivity spectrum ω_d , and the Ioffe-Regel frequency ω_{IR} are quantitatively identical.

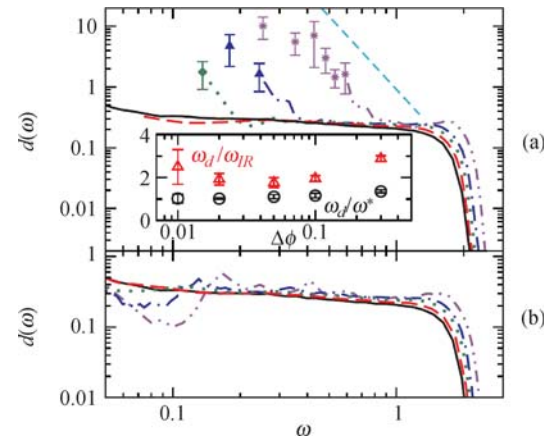


Fig. 14 Diffusivity $d(\omega)$ for the 3-D (a) unstressed and (b) stressed bi-disperse ($R = 1.4$) systems composed of $N = 2000$ spheres interacting via harmonic repulsion at $\Delta\phi = 10^{-6}$ (solid black line), 0.01 (red dashed line), 0.05 (green dotted line), 0.1 (blue dot-dashed line), and 0.3 (purple dot-dot-dashed line). In (a), the cyan dashed line indicates a power-law of ω^{-4} and the closed symbols indicate the degenerate sets of discrete plane-wave modes in our finite system. Inset: The ratios ω_d/ω^* (open circles) and ω_d/ω_{IR} (open triangles) versus $\Delta\phi$. Adapted from Refs. [78, 79], Copyright © 2009, 2010 the American Physical Society.

In this section, we have focused on a class of low-frequency modes of vibration that uniquely exist in amorphous solids and may play the key role in properties of glasses. These modes are excessive beyond Debye's prediction and thus form the boson peak. In our simple model of jammed solids, they accumulate at the boson peak frequency and show multiple crossovers, e.g. onset of the flat density of states, crossover between quasi-localization and extensive modes with flat participation ratio, and Ioffe-Regel limit. Furthermore, these modes are strongly anharmonic with the lowest energy

barrier upon mechanical failure. It is important to have a clear picture of these modes, e.g. the origin of quasi-localization and their response to the thermal excitation or deformation, before we are able to have a better understanding of the glass transition and properties of amorphous solids.

7 Universality of the glass transition

In the sections above, we have discussed properties of amorphous systems in the jammed region of the jamming phase diagram (see Fig. 1). Approaching jamming from the unjammed region, amorphous systems undergo the rapid slowdown of dynamics [10, 11, 110–112]. Glass transition occurs when the viscosity or the relaxation time exceeds the time window that can be measured. From the jamming phase diagram, we can see that there are two typical routes of glass transition: molecular glass transition via decreasing the temperature [9–11] and colloidal glass transition via increasing the density (or pressure) [1, 110, 113]. Although they show similar abrupt increase of the viscosity or relaxation time with a small change of temperature or density, these two glass transitions have been treated as two different processes, because molecular glass transition and colloidal glass transition seem to be controlled by energy and entropy, respectively. However, their similarities imply that there may be some common underlying physics for both transitions.

As shown in Fig. 15(a) and (b), the relaxation time of unjammed liquids strongly depends on the pressure during molecular glass transition and the temperature during colloidal glass transition. However, when we plot the rescaled relaxation time $\tau/\sqrt{m/(p\sigma)}$ versus $T/(p\sigma^3)$, all the relaxation time curves measured at low temperatures or pressures in the vicinity of Point J collapse onto the same master curve [114],

$$\tau/\sqrt{m/(p\sigma)} = f(T/(p\sigma^3)) \quad (24)$$

Surprisingly, this scaling collapse does not show any particle interaction dependence for all the repulsions defined in Section 2 as well as hard spheres. This robust result demonstrates that the glass transition is not solely controlled by either the temperature or the pressure. It is the ratio of the temperature to the pressure that really governs the dynamics of glass-forming liquids. Therefore, the molecular glass transition and colloidal glass transition are actually equivalent in the vicinity of Point J.

It is a long-standing question whether there exists a thermodynamic glass transition. The main difficulty is the inaccessibility of the infinite relaxation time or viscosity, which makes it impossible to locate the glass transition accurately. In practice, the glass transition

is always estimated from the functional fit to the relaxation time data that we can have. Typical fits include the mode coupling form [115], Vogel–Fulcher function, and Elmatad–Chandler–Garrahan function [116]. In Fig. 15(b), the scaling collapse is fitted by both the Vogel–Fulcher and Elmatad–Chandler–Garrahan equations. Both equations fit the data equally well in the regime where the rescaled relaxation time diverges quickly. From the functional form of both equations (see the caption to Fig. 15), we can see that Vogel–Fulcher equation predicts a glass transition at $T/p > 0$, while Elmatad–Chandler–Garrahan equation implies that glass transition happens at $T/p = 0$. It is thus still ambiguous whether the glass transition happens at $T > 0$ or not. Consequently, there remain elusive open questions such as whether the hard sphere colloidal glass transition happens at Point J and whether and how Point J affects the glass transition.

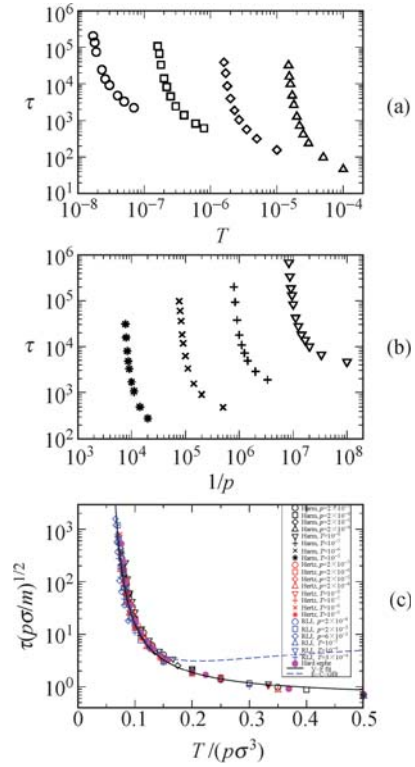


Fig. 15 (a) Relaxation time τ versus temperature T at different fixed pressures: $p = 10^7$ (circles), $p = 2 \times 10^6$ (squares), $p = 2 \times 10^5$ (diamonds), and $p = 2 \times 10^4$ (upward triangles). (b) Relaxation time τ versus inverse pressure $1/p$ at different fixed temperatures: $T = 10^8$ (downward triangles), $T = 10^7$ (pluses), $T = 10^6$ (crosses), and $T = 10^5$ (stars). In (a) and (b) systems consist of 1000 particles in 3-D interacting via harmonic repulsion. (c) Scaled relaxation time, $\tau/\sqrt{m/(p\sigma)}$, versus scaled temperature $T/(p\sigma^3)$ for all the data for the harmonic potential (black) in (a) and (b) as well as for the Hertzian potential (red), the hard-sphere potential (magenta), and the Weeks–Chandler–Andersen potential (blue). Black solid curve is the Vogel–Fulcher fit: $y = 0.59\exp[0.18/(x - 0.045)]$, where $y = \tau/\sqrt{m/(p\sigma)}$ and $x = T/(p\sigma^3)$. Blue-dashed curve is a fit to the Elmatad–Chandler–Garrahan form: $y = 3.1\exp[0.064(x^{-1} - 4.72)^2]$. Adapted from Ref. [114], Copyright © 2009 the American Physical Society.

8 Conclusions

Jamming has been extensively studied in the last decade since the proposal of the jamming phase diagram. In this paper, we reviewed recent progress on the study of jamming, focusing on the physics near Point J mainly from computational work. As we have shown, Point J not only is an intriguing critical point full of new physics but also provides us with great implications on properties of a variety of amorphous systems and long-standing issue of the glass transition. There are still lots of unknowns about jamming near Point J , for instance, theory of the critical phenomena at Point J that may be beyond our knowledge of the conventional phase transitions, the origin of the low-frequency quasi-localized modes and connections to the glass transition, the extension of the jamming transition to nonzero temperature and shear stress, the correct relationship between the jamming transition and glass transition, etc.

Furthermore, a large portion of published studies deal with idealized models ignoring influences from long-range attraction, friction, aspherical particle shape, etc. Although there have been some approaches with consideration of these factors [7, 50, 52, 57, 59–61, 117, 118], we still lack knowledge on their effects on what we have learned from ideal spherical systems. The study in this direction would be of more general interests because the systems are more realistic.

Due to the complication of the amorphous systems caused by the structural randomness and difficulties to probe certain quantities such as the normal modes of vibration in real experimental systems, experimental work on jamming lags behind numerical simulations. It is encouraging that recent experiments on granular materials [68, 76, 119, 120] and colloidal suspensions [8, 121–124] have reproduced some results from ideal numerical models, although there still remain a lot to do. Among all the typical amorphous systems mentioned in the beginning of this paper, colloidal suspensions, especially those with soft potentials, are the most ideal experimental systems to study jamming, particularly glass transition and vibrational properties. Recent colloidal experiments have demonstrated such advantages. It is thus hopeful to expect breakthroughs of experimental study of jamming in colloidal experiments.

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