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Research article

Conformable mass, momentum, energy, and Galilean conservation laws of the conformable modified Korteweg-de Vries equation

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Abstract: We investigate the conservation laws of the conformable fractional modified Korteweg–de Vries (mKdV) equation, in which the temporal derivative of order α and the spatial derivatives of orders $\beta_0, \beta_1, \beta_2, \beta_3$ are all taken in the conformable sense. Our main result provides a complete classification of all conservation laws of this equation whose characteristics have order at most 3. We show that the four most fundamental conservation laws of the classical mKdV equation, i.e., the mass, momentum, energy, and Galilean conservation laws, admit conformable counterparts, and we determine the precise conditions on the spatial orders $\beta_0, \beta_1, \beta_2, \beta_3$ under which each counterpart holds. We also establish that, under these conditions, the conformable conservation laws provide a complete classification in the same sense as the classical case. The conserved densities and fluxes are written out explicitly, and their physical interpretations as conformable mass, momentum, energy, and centre of momentum are discussed.

Keywords: conformable modified KdV equation; conservation law
Mathematics Subject Classification: 35B06, 35Q53

1. Introduction

The modified Korteweg-de Vries (mKdV) equation

$$\frac{\partial u}{\partial t} + 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

is one of the most important nonlinear dispersive equations in mathematical physics. Besides its significance in the study of nonlinear wave propagation, the mKdV equation possesses rich geometric

and analytical structures. In particular, the equation admits infinitely many independent conservation laws up to equivalence [1–3]. The four independent conservation laws with the lowest order, which are usually interpreted as the mass, momentum, energy, and Galilean conservation laws, respectively, have very clear physical meanings, so they are the most widely used in practice and can be considered as the four most important conservation laws. From a computational perspective, these four conservation laws provide a complete classification of the mKdV equation's conservation laws whose orders of characteristics are no higher than 3 (the order of characteristic used here will be defined in §2.2), meaning that any conservation law of the mKdV equation whose order of characteristic is no higher than 3 is equivalent to a linear combination of these four conservation laws.

Since its introduction in 2014, conformable fractional calculus has attracted considerable attention. Introduced by Khalil et al. in [4] and further developed by Abdeljawad in [5], the conformable derivative provides a fractional differentiation framework that preserves many fundamental properties of the classical derivative, including linearity, product rules, quotient rules, and a simple local representation for differentiable functions. Consequently, conformable derivatives have been applied to a wide variety of ordinary and partial differential equations, where they offer a comparatively tractable framework for investigating fractional-order phenomena while retaining many tools from classical differential equations.

Motivated by these developments, we ask whether the fundamental geometric structures of the classical mKdV equation persist under conformable fractionalisation, and more specifically,

(Q1) Do the four most important conservation laws of the mKdV equation remain valid after conformable fractionalisation?

(Q2) Are the conditions for them to remain valid the same?

(Q3) Do the conservation laws remaining valid still provide a complete classification of all conservation laws whose orders of characteristics are no higher than 3?

(Q4) After conformable fractionalisation, do the conservation laws have similar physical interpretations?

To address these questions, we study the conformable fractional-order mKdV equation

$$\frac{\partial^{\alpha} u}{\partial t} + 6u^2 \frac{\partial^{\beta_0} u}{\partial x} + \frac{\partial^{\beta_3}}{\partial x}\frac{\partial^{\beta_2}}{\partial x}\frac{\partial^{\beta_1} u}{\partial x} = 0, \tag{1.2}$$

where $\frac{\partial^{\alpha}}{\partial t}$ and $\frac{\partial^{\beta_i}}{\partial x}$ stand for the conformable partial derivatives of order α and β_i of u with respect to t and x, respectively, and $\frac{\partial^{\beta_3}}{\partial x}\frac{\partial^{\beta_2}}{\partial x}\frac{\partial^{\beta_1}}{\partial x}$ stands for the right-to-left composition of conformable derivative operators. We consider $\alpha \in (0, 1]$ as the temporal order of the conformable derivative and $\beta_0, \beta_1, \beta_2, \beta_3 \in (0, 1]$ as spatial orders of conformable derivatives. Through the rest of this paper, we denote

$$\beta = \beta_1 + \beta_2 + \beta_3. \tag{1.3}$$

The definitions of the conformable derivative and conservation law will be introduced in §2.

Our main theorem and its remarks provide an answer for the questions (Q1), (Q2), and (Q3) mentioned above, and we will discuss the question (Q4) in §2.3.

Theorem 1.1. *We write*

$$
\begin{aligned}
P_1 =& x^{\beta_0-1}u,\\
Q_1 =& t^{\alpha-1}\Big\{2u^3 + x^{\beta_0-\beta}\big[(\beta_0-\beta_3)(\beta_0-\beta_2-\beta_3)u + x^2u_{xx} - (\beta_0+\beta_1-\beta_3-1)xu_x\big]\Big\},\\
P_2 =& \frac{1}{2}x^{\beta_0-1}u^2,\\
Q_2 =& \frac{1}{2}t^{\alpha-1}\Big\{3u^4 - x^{2-2\beta_1}u_x^2 + 2x^{1-2\beta_1}u[(1-\beta_1)u_x + xu_{xx}]\Big\},\\
P_3 =& \frac{1}{2}x^{\beta_0-1}u^4 - \frac{1}{2}x^{1-\beta_1}u_x^2,\\
Q_3 =& \frac{1}{2}x^{-2(\beta_0+\beta_1)}\Big\{t^{\alpha-1}[2u^3x^{\beta_0+\beta_1} + (1-\beta_1)xu_x + x^2u_{xx}]^2 + 2x^{2\beta_0+\beta_1+1}u_xu_t\Big\},\\
P_4 =& \frac{1}{2}\beta_0x^{\beta_0-1}t^\alpha u^4 - \frac{1}{6}\alpha x^{2\beta_0-1}u^2 - \frac{1}{2}\beta_0x^{1-\beta_0}t^\alpha u_x^2,\\
Q_4 =& \frac{1}{6}x^{-4\beta_0}t^{\alpha-1}\Big\{3\beta_0t^\alpha[2x^{2\beta_0}u^3 + (1-\beta_0)xu_x + x^2u_{xx}]^2\\
&+ x^{3\beta_0}[xu_x(6\beta_0tu_t + \alpha xu_x) - 3\alpha x^{2\beta_0}u^4 - 2\alpha xu((1-2\beta_0)u_x + xu_{xx})]\Big\}.
\end{aligned}
$$

Then the linear combinations of the (P_i, Q_i)*'s* ($i = 1, 2, 3, 4$) *and their equivalent forms are all possible conservation laws of Eq* (1.2) *with a characteristic of the form* $R(x, t, u, u_x, u_{xx}, u_{xxx})$ *or* $R(x, t, u, u_x, u_t)$. *More specifically, we have*

(1) *the pair* (P_1, Q_1) *is a conservation law of* (1.2) *if and only if*

$$\beta_0 = \beta_1 + \beta_2 + \beta_3 \quad or \quad \beta_0 = \beta_2 + \beta_3 \quad or \quad \beta_0 = \beta_3; \tag{C1}$$

(2) *the pair* (P_2, Q_2) *is a conservation law of* (1.2) *if and only if*

$$\beta_0 = \beta_3 \quad and \quad \beta_1 = \beta_2; \tag{C2}$$

(3) *the pair* (P_3, Q_3) *is a conservation law of* (1.2) *if and only if*

$$\beta_0 = \beta_2 = \beta_3; \tag{C3}$$

(4) *the pair* (P_4, Q_4) *is a conservation law of* (1.2) *if and only if*

$$\beta_0 = \beta_1 = \beta_2 = \beta_3; \tag{C4}$$

(5) *if none of the conditions* (C1)–(C4) *hold, then Eq* (1.2) *does not have any non-trivial conservation law with a characteristic of the form* $R(x, t, u, u_x, u_{xx}, u_{xxx})$ *or* $R(x, t, u, u_x, u_t)$.

Remark 1.2 (An equivalent restatement of Theorem 1.1)**.** Suppose that (P, Q) is a conservation law of (1.2) whose characteristic is of the form $R(x, t, u, u_x, u_{xx}, u_{xxx})$ or $R(x, t, u, u_x, u_t)$.

(1) If $\beta_0 = \beta_1 + \beta_2 + \beta_3$ or $\beta_0 = \beta_2 + \beta_3$ or ($\beta_0 = \beta_3$ and $\beta_0 \neq \beta_2$ and $\beta_1 \neq \beta_2$), then (P, Q) is equivalent to a conservation law (P', Q') with

$$(P', Q') = c_1(P_1, Q_1), \quad c_1 \in \mathbb{R}.$$

(2) If $\beta_0 = \beta_3$ and $\beta_1 = \beta_2$ and $\beta_0 \neq \beta_1$, then (P, Q) is equivalent to a conservation law (P', Q') with

$$(P', Q') = c_1(P_1, Q_1) + c_2(P_2, Q_2), \quad c_1, c_2 \in \mathbb{R}.$$

(3) If $\beta_0 = \beta_2 = \beta_3$ and $\beta_0 \neq \beta_1$, then (P, Q) is equivalent to a conservation law (P', Q') with

$$(P', Q') = c_1(P_1, Q_1) + c_3(P_3, Q_3), \quad c_1, c_3 \in \mathbb{R}.$$

(4) If $\beta_0 = \beta_1 = \beta_2 = \beta_3$, then (P, Q) is equivalent to a conservation law (P', Q') with

$$(P', Q') = c_1(P_1, Q_1) + c_2(P_2, Q_2) + c_3(P_3, Q_3) + c_4(P_4, Q_4), \quad c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

(5) Otherwise, i.e., if $\beta_0 \neq \beta_1+\beta_2+\beta_3$ and $\beta_0 \neq \beta_2+\beta_3$ and $\beta_0 \neq \beta_3$, then (P, Q) is trivial, i.e., equivalent to $(0, 0)$.

Remark 1.3. (1) The logical relationships among conditions (C1)–(C4) are illustrated in Figure 1.

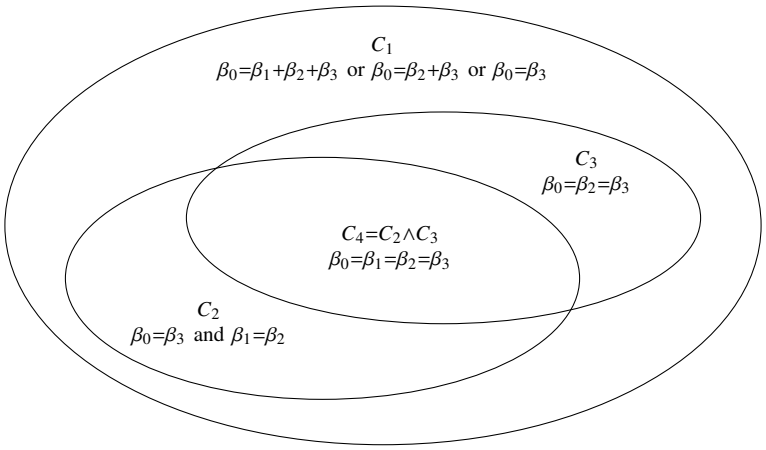


Figure 1. Relation of the conditions (C1)–(C4).

(2) By taking $\alpha = \beta_0 = \beta_1 = \beta_2 = \beta_3 = 1$, we see that the conservation laws (P_1, Q_1), (P_2, Q_2), (P_3, Q_3), and (P_4, Q_4) can be regarded as conformable counterparts of the classical mass, momentum, energy, and Galilean conservation laws for the integer-order mKdV equation.

2. Preliminaries and further discussions

2.1. Conformable derivative

The conformable derivative is a fractional derivative introduced by Khalil et al. in [4] in 2014:

Definition 2.1. [4] Let $f : (0, +\infty) \to \mathbb{R}$ be a function, $\alpha \in (0, 1]$, and $t \in (0, +\infty)$. If the following limit exists, then it is called the conformable fractional derivative of order α of f at t and denoted by $\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t}$ or $\frac{\mathrm{d}^\alpha}{\mathrm{d}t} f(t)$:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t} = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

In this case, f is said to be conformably differentiable of order α at t.

Note that $\frac{\mathrm{d}^1 f(t)}{\mathrm{d}t}$ coincides with the classical definition of first derivative $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = f'(t)$. In general, it does not hold that

$$\frac{\mathrm{d}^{\alpha+\beta} f(t)}{\mathrm{d}t} = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}^{\beta} f(t)}{\mathrm{d}t}.$$

For this reason, we do not write

$$\frac{\partial^{\beta_1+\beta_2+\beta_3} u}{\partial x}$$

for the third term in the left-hand side of (1.2). On the other hand, the conformable fractional derivative satisfies the constant rule, sum rule, product rule, and quotient rule as the classical derivative:

Lemma 2.2. *[4] Let $\alpha \in (0, 1]$, $t \in (0, +\infty)$, $a, b, c \in \mathbb{R}$, and f and g be functions conformably differentiable of order α at t. Then we have*

(1) $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} c = 0$;

(2) $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} t^c = ct^{c-\alpha}$;

(3) $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t}(af + bg)(t) = a\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} f(t) + b\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} g(t)$;

(4) $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t}(fg)(t) = g(t)\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} f(t) + f(t)\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} g(t)$;

(5) $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t}(f/g)(t) = \left(g(t)\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} f(t) - f(t)\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} g(t)\right)/g(t)^2$.

Lemma 2.3. *[6, 7] Let $\alpha \in (0, 1]$, $t \in (0, +\infty)$, and f be a function defined on a neighbourhood of t. Then the conformable differentiability of f at t is equivalent to the classical differentiability of f at t. In this case, we have $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} f(t) = t^{1-\alpha} f'(t)$, where $f'(t)$ denotes classical derivative of f.*

Since its introduction, conformable fractional calculus has generated extensive research activity across multiple directions. On the theoretical side, its algebraic structure—including chain rules, integration by parts, and Laplace transforms—was systematically developed by [5], while the properties of the conformable derivative were further examined and extended in [8, 9]. In addition, several recent studies have revisited analytical properties, structural extensions, and solution techniques within the conformable framework, including new results on conformable differential systems and associated functional equations [10–12].

The conformable derivative has been extensively applied in the study of partial differential equations, owing precisely to its preservation of classical calculus rules. Typical examples include conformable heat equations [13, 14], Burgers-type equations [15, 16], Klein-Gordon equations [17], advection-diffusion equations [18, 19], and Navier-Stokes equations [20]. More recently, conformable formulations of nonlinear evolutionary equations have continued to attract attention, with new investigations addressing both analytical and wave-structure properties in generalized settings [21].

In the context of nonlinear dispersive equations related to KdV and mKdV, conformable fractional KdV-type models have been studied from several perspectives: Travelling wave and soliton solutions have been obtained via a variety of analytic methods [22–24], and ion-acoustic wave propagation in plasmas has been modelled using the conformable mKdV framework [25]. Recent developments also include refined analytical treatments of wave dynamics and solution structures for conformable dispersive systems [26]. Recent research on the long-time asymptotic behavior of mKdV-type equations can be seen in [27–29], while research on the construction and characterization of soliton and other explicit solutions can be seen in [30, 31].

Symmetry-based approaches have also been applied to construct conservation laws and group-invariant solutions of conformable partial differential equations more broadly [32–34]; conservation laws of conformable heat equations and Burgers equations were studied in [35, 36].

Beyond purely analytical investigations, there is a growing body of work demonstrating that conformable derivative models can provide good agreement with measured data in physically motivated settings. In the modelling of non-Darcian flow and anomalous diffusion in low-permeability porous media, Zhou et al. [37, 38] have showed that conformable derivative diffusion models fit experimental data on chloride ion transport in reinforced concrete significantly better than standard integer-order models. Related conformable advection-dispersion models for contaminant migration were validated against laboratory transport data in porous media [18, 39]. In heat transfer, Rosales-García et al. [40] found that Newton's law of cooling expressed via the conformable derivative yields a stretched-exponential temperature profile that matches experimental cooling data significantly more accurately than the classical exponential model, a conclusion supported by subsequent work using generalised conformable kernels [41]. In materials science, conformable derivative Maxwell and Zener viscoelastic models have been found to reproduce experimentally observed relaxation moduli and creep compliance with improved fidelity [42]. In the geophysical modelling of pollutant diffusion in soil, Kim et al. [43] showed that the conformable fractional time derivative naturally captures the anomalous sub-diffusive behaviour observed in inhomogeneous porous materials and proposed a method to calibrate the fractional order against field measurements. These investigations indicate that the conformable framework provides a tractable setting for incorporating fractional-order effects in physical models, even when its interpretation as a genuine fractional calculus remains a matter of ongoing discussion.

2.2. Conservation law

Let X and U denote $\mathbb{R}^2$ and $\mathbb{R}^1$ with coordinate system (x, t) and (u), respectively. For an open subset $M \subseteq X \times U$, we write $M^{(n)}$ for the jet space of M with order n. A pair (P, Q) of functions on $M^{(n)}$, i.e., on the variables $x, t, u, u_x, u_t, u_{xx}, \ldots$, is called a conservation law of Eq (1.2) if the restriction of the divergence expression

$$\mathrm{D}_t P + \mathrm{D}_x Q$$

vanishes on any smooth solution of (1.2), where D_t and D_x denote the total derivatives on t and x, respectively. In this case, P and $-Q$ are called the conserved density and flux of the conservation law, respectively. We take two subintervals $[t_0, t_1]$ and $[x_0, x_1]$ of $(0, +\infty)$. For any conservation law (P, Q) of (1.2), we have

$$\int_{x_0}^{x_1} P|_{t=t_1}\,\mathrm{d}x - \int_{x_0}^{x_1} P|_{t=t_0}\,\mathrm{d}x = \int_{t_0}^{t_1} (-Q)|_{x=x_1}\,\mathrm{d}t - \int_{t_0}^{t_1} (-Q)|_{x=x_0}\,\mathrm{d}t.$$

In particular, if $Q|_{x=x_i} \equiv 0$ for $i = 0$ and 1, then the value $\int_{x_0}^{x_1} P\,\mathrm{d}x$ is a constant independent of t.

Note that if both (P, Q) and (P', Q') are conservation laws of (1.2), then so is $(\lambda P + \mu P', \lambda Q + \mu Q')$ for any constants λ and μ. We say that a conservation law (P, Q) is trivial if P and Q both vanish on any smooth solution of (1.2), or if $\mathrm{D}_t P + \mathrm{D}_x Q$ vanishes identically (not merely on solutions), or if (P, Q) is a linear combination of the above two cases. Two conservation laws (P, Q) and (P', Q') are called equivalent if $(P - P', Q - Q')$ is trivial. Generally, we are only interested in studying conservation laws up to equivalence.

Suppose (P, Q) is a conservation law of (1.2). Then there exists an equivalent conservation law (P', Q') such that

$$\mathrm{D}_t P' + \mathrm{D}_x Q' = R\,\Delta,$$

where Δ is the equation defined in the jet space determined by the origin equation (1.2) (also see (3.2) below) and R is a smooth function defined on the jet space. We call R the characteristic of the conservation law (P, Q).

For the conformable mKdV equation, as well as the classical mKdV equation which can be regarded as the special case of conformable mKdV equation when $\alpha = \beta_0 = \beta_1 = \beta_2 = 1$, every conservation law has a unique equivalent characteristic of the following form:

$$R(x, t, u, u_x, u_{xx}, \ldots, u_n), \quad \frac{\partial R}{\partial u_n} \neq 0, \quad u_n = u_{\underbrace{x \ldots x}_{n}}$$

for some integer $n \geq -1$. (When $n = -1$, the formula above reads $R = R(x, t)$; when $n = 0$, the formula above reads $R = R(x, t, u)$ with $\partial R/\partial u \neq 0$.) In this case, we say (P, Q) has a characteristic of order n. For instance, if a conservation law (P, Q) has a characteristic of the form $R(x, t, u, u_x, u_t)$ with $\partial R/\partial u_t \neq 0$, then it has a characteristic of order 3, namely, it is equivalent to a conservation law (P', Q') whose characteristic has the form $R'(x, t, u, u_x, u_{xx}, u_{xxx})$ with $\partial R'/\partial u_{xxx} \neq 0$.

As we mentioned above, in most applications of the classical mKdV equation, people only use the conservation laws whose characteristic orders are at most 3. In fact, the mass conservation law of mKdV equation has a characteristic of order -1, the momentum conservation law has that of order 0, the energy conservation law has that of order 2, and there does not exists a conservation law having a characteristic of order 1 or 3.

Conservation laws provide fundamental information about the intrinsic structure of differential equations and play a central role in mathematical physics, geometric analysis, and numerical computation. In the classical setting, systematic approaches to conservation laws have been developed through the works of Anco, Bluman, and others [44, 45]. For classical nonlinear evolution equations such as the KdV equation, the modified mKdV equation, and the generalised KdV equation, conservation laws are closely connected with integrability, stability theory, and long-time dynamics. Consequently, the determination and classification of conservation laws have long been a central topic in the analysis of nonlinear partial differential equations.

For conformable differential equations, however, the corresponding theory is still developing. For instance, the conservation laws of conformable heat equations and conformable Burgers equations are studied in [35, 36].

2.3. Further discussions on the main theorem

Let D be a nonempty open subset of $\mathbb{R}$, and let $\mathbb{R}^+ = (0, +\infty)$. We consider a conformable time-fractional differential evolution equation in the general form

$$\partial_t^\alpha u(x, t) = f(x), \quad (x, t) \in D \times \mathbb{R}^+ \tag{2.1}$$

where $f(x)$ is a smooth function defined on D. By referring to [46], we introduce a new variable

$$\tau = \frac{t^\alpha}{\alpha}. \tag{2.2}$$

The inverse map of (2.2) is given by

$$t = (\alpha\tau)^{\frac{1}{\alpha}}, \tag{2.3}$$

so (x, τ) is another coordinate system of $D \times \mathbb{R}^+$. Since $\mathrm{d}\tau/\,\mathrm{d}t = t^{\alpha-1}$, we have

$$t^{1-\alpha}\frac{\partial u}{\partial t}(x,t) = \frac{\partial u}{\partial \tau}(x,\tau).$$

In particular, we see from Lemma 2.3 that under the coordinate system (x, τ), Eq (2.1) can be rewritten as an equivalent form with temporal order 1:

$$\frac{\partial u}{\partial \tau}(x,\tau) = f(x), \quad (x,\tau) \in D \times \mathbb{R}^+. \tag{2.4}$$

Suppose that $(\tilde{P}, \tilde{Q})$ is an arbitrary conservation law of (2.4) with characteristic $\tilde{R}$. Then by letting $P = \tilde{P}$, $Q = t^{\alpha-1}\tilde{Q}$, and $R = t^{\alpha-1}\tilde{R}$ after substituting (2.2), we obtain a conservation law (P, Q) of (2.1) with characteristic R. This is a bijective correspondence between the conservation-law space of (2.1) and that of (2.4), with the inverse obtained from (2.3). For the conformable mKdV equation (1.2) discussed in this paper, this correspondence also prevents the order of a conservation law. In particular, it follows before any detailed coefficient calculation that the following properties of (1.2) cannot depend on α:

- the existence or nonexistence of local conservation laws;
- the dimension of the conservation-law space;
- the spatial differential order of their characteristics; and
- the conditions on the spatial orders $\beta_0, \beta_1, \beta_2, \beta_3$ under which they exist.

This is in complete agreement with our main theorem.

Moreover, under the condition (C4), i.e., $\beta_0 = \beta_1 = \beta_2 = \beta_3$, we introduce a new variable

$$\sigma = \frac{x^{\beta_0}}{\beta_0}. \tag{2.5}$$

Then (σ, τ) forms a new coordinate system of $\mathbb{R}^+ \times \mathbb{R}^+$, and under this Eq (1.2) can be rewritten as the classical mKdV equation

$$\frac{\partial u}{\partial \tau} + 6u^2\frac{\partial u}{\partial \sigma} + \frac{\partial^3 u}{\partial \sigma^3} = 0, \quad (\sigma,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.6}$$

Hence, (C4) is the sufficient condition for that all of (P_i, Q_i) $(i = 1, 2, 3, 4)$ are conservation laws of (1.2), and our main theorem shows that it is also the necessary condition.

In the rest of this subsection, we briefly discuss the physical interpretation of the conformable conserved quantities and try to answer the question (Q4) at the beginning of this paper. Since the physical applicability of Eq (1.2) remains unclear, the following discussion should be understood only as a heuristic interpretation of the conservation laws.

Let $x_1 > x_0 > 0$ and $t_1 > t_0 > 0$. For a smooth function $u(x, t)$ defined on a neighbourhood W of $[x_0, x_1] \times [t_0, t_1] \subseteq \mathbb{R}^2$, we regard $u(x, t)$ as a scalar quantity transported in a hypothetical one-dimensional medium. By fixing the smooth function $u(x, t)$, the P_i's and Q_i's defined in Theorem 1.1 become functions on x and t depending on the parameters $\alpha, \beta_0, \beta_1, \beta_2$, and β_3.

Definition 2.4. Let $x_1 > x_0 > 0$ and $t_1 > t_0 > 0$. For a fixed smooth function $u(x,t)$ defined on a neighbourhood W of $[x_0,x_1]\times[t_0,t_1]$, we write the integrals of $P_1(x,t)$, $P_2(x,t)$, $P_3(x,t)$, and $P_4(x,t)$ with respect to x on the interval $[x_0,x_1]$ by $M(t)$, $P(t)$, $E(t)$, and $G(t)$, respectively. That is,

$$\begin{aligned}
M(t) &= \int_{x_0}^{x_1} x^{\beta_0-1}u(x,t)\,\mathrm{d}x,\\
P(t) &= \int_{x_0}^{x_1} \frac{1}{2}x^{\beta_0-1}u(x,t)^2\,\mathrm{d}x,\\
E(t) &= \int_{x_0}^{x_1} \frac{1}{2}x^{\beta_0-1}u(x,t)^4 - \frac{1}{2}x^{1-\beta_1}u_x(x,t)^2\,\mathrm{d}x,\\
G(t) &= \int_{x_0}^{x_1} \frac{1}{2}\beta_0 x^{\beta_0-1}t^{\alpha}u(x,t)^4 - \frac{1}{6}\alpha x^{2\beta_0-1}u(x,t)^2 - \frac{1}{2}\beta_0 x^{1-\beta_0}t^{\alpha}u_x(x,t)^2\,\mathrm{d}x.
\end{aligned}$$

We also write

$$C(t) = \frac{\int_{x_0}^{x_1}\beta_0^{-1}x^{\beta_0}P_2(x,t)\,\mathrm{d}x}{\int_{x_0}^{x_1}P_2(x,t)\,\mathrm{d}x} = \frac{\int_{x_0}^{x_1}\frac{1}{2}\beta_0^{-1}x^{2\beta_0-1}u(x,t)^2\,\mathrm{d}x}{\int_{x_0}^{x_1}\frac{1}{2}x^{\beta_0-1}u(x,t)^2\,\mathrm{d}x}.$$

We call $M(t)$, $P(t)$, $E(t)$, and $C(t)$ the β_0-order conformable mass, β_0-order conformable momentum, (β_0,β_1)-order conformable energy, and β_0-order conformable centre of momentum of $u(x,t)$ on $[x_0,x_1]$ at the time t, respectively.

Corollary 2.5. *Let $u(x,t)$ be an arbitrary smooth solution of* (1.2) *defined on a neighbourhood W of $[x_0,x_1]\times[t_0,t_1]\subseteq\mathbb{R}^2$.*

(1) *(Conformable mass conservation law) If*

$$\beta_0 = \beta_1+\beta_2+\beta_3 \quad or \quad \beta_0 = \beta_2+\beta_3 \quad or \quad \beta_0 = \beta_3, \tag{C1}$$

then for the β_0-order conformable mass $M(t)$ of $u(x,t)$, we have

$$M(t) - M(t_0) = \int_{t_0}^{t} Q_1(x_1,s)\,\mathrm{d}s - \int_{t_0}^{t} Q_1(x_0,s)\,\mathrm{d}s$$

for any $t\in[t_0,t_1]$. In particular, under the condition (C1)*, if*

$$u(x_0,t) = u(x_1,t),\ u_x(x_0,t) = u_x(x_1,t),\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

or

$$\beta_0+\beta_1-\beta_3 = 1,\ u(x_0,t) = u(x_1,t),\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

for any $t\in[t_0,t_1]$, then the β_0-order conformable mass $M(t)$ is a constant, independent of t.

(2) *(Conformable momentum conservation law) If*

$$\beta_0 = \beta_3 \quad and \quad \beta_1 = \beta_2, \tag{C2}$$

then for the β_0-order conformable momentum $P(t)$ of $u(x,t)$, we have

$$P(t) - P(t_0) = \int_{t_0}^{t} Q_2(x_1, s)\,\mathrm{d}s - \int_{t_0}^{t} Q_2(x_0, s)\,\mathrm{d}s$$

for any $t \in [t_0, t_1]$. In particular, under the condition (C2)*, if*

$$u(x_0,t) = u(x_1,t),\ u_x(x_0,t) = u_x(x_1,t),\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

for any $t \in [t_0, t_1]$, then the β_0-order conformable momentum $P(t)$ is a constant, independent of t.

(3) *(Conformable energy conservation law) If*

$$\beta_0 = \beta_2 = \beta_3, \tag{C3}$$

then for the (β_0, β_1)-order conformable energy $E(t)$ of $u(x,t)$, we have

$$E(t) - E(t_0) = \int_{t_0}^{t} Q_3(x_1, s)\,\mathrm{d}s - \int_{t_0}^{t} Q_3(x_0, s)\,\mathrm{d}s$$

for any $t \in [t_0, t_1]$. In particular, under the condition (C4)*, if*

$$u(x_0,t) = u(x_1,t),\ u_x(x_0,t) = u_x(x_1,t),\ u_t(x_0,t) = u_t(x_1,t),\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

or

$$\beta_1 = 1,\ u(x_0,t) = u(x_1,t),\ u_t(x_0,t) = u_t(x_1,t) = 0,\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

for any $t \in [t_0, t_1]$, then the (β_0, β_1)-order conformable energy $E(t)$ is a constant, independent of t.

(4) *(Conformable Galilean conservation law) If*

$$\beta_0 = \beta_1 = \beta_2 = \beta_3, \tag{C4}$$

then we have

$$G(t) - G(t_0) = \int_{t_0}^{t} Q_4(x_1, s)\,\mathrm{d}s - \int_{t_0}^{t} Q_4(x_0, s)\,\mathrm{d}s$$

for any $t \in [t_0, t_1]$. In particular, under the condition (C4)*, if*

$$u(x_0,t) = u(x_1,t),\ u_x(x_0,t) = u_x(x_1,t),\ u_t(x_0,t) = u_t(x_1,t),\ u_{xx}(x_0,t) = u_{xx}(x_1,t)$$

for any $t \in [t_0, t_1]$, then the α-order conformable derivative $\frac{\mathrm{d}^\alpha}{\mathrm{d}t} C(t)$ of the β_0-order conformable centre of momentum $C(t)$ is a constant, independent of t.

Proof. Only the latter part of conclusion (4) cannot be seen directly from the definition of conservation law. Under the conditions of the latter part of conclusion (4), $G = G(t)$ is a constant, and we see from conclusions (2) and (3) that the conformable momentum $P = P(t)$ and conformable energy $E = E(t)$ of $u(x,t)$ are also both constants. Hence, we deduce that

$$\frac{G}{P} = \Big[\int_{x_0}^{x_1} \frac{1}{2}\beta_0 x^{\beta_0 - 1} t^\alpha u(x,t)^4 - \frac{1}{6}\alpha x^{2\beta_0 - 1} u(x,t)^2 - \frac{1}{2}\beta_0 x^{1-\beta_0} t^\alpha u_x(x,t)^2\,\mathrm{d}x \Big] / P$$

$$=\Big[t^{\alpha}\beta_0 E - \frac{1}{3}\alpha\beta_0 \int_{x_0}^{x_1} \frac{1}{2}\beta_0^{-1} x^{2\beta_0-1} u(x,t)^2 \,\mathrm{d}x\Big]/P = t^{\alpha}\beta_0 \frac{E}{P} - \frac{1}{3}\alpha\beta_0 C(t).$$

By differentiating with respect to t, we obtain

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t} C(t) = t^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} C(t) = \frac{3E}{P},$$

which completes the proof. □

Assume that condition (C1) holds. Let $u(x,t)$ be an arbitrary smooth solution of (1.2) defined on a neighbourhood W of $[x_0, x_1] \times [t_0, t_1] \subseteq \mathbb{R}^2$. We regard $u(x,t)$ as a scalar quantity transported in a hypothetical one-dimensional medium. The conservation law (P_1, Q_1) can then be interpreted as a balance law for the total amount of this quantity.

When $\alpha = \beta_0 = \beta_1 = \beta_2 = \beta_3 = 1$, Eq (1.2) reduces to the classical integer-order mKdV equation, and Part (1) of Corollary 2.5 becomes the usual conservation law of mass. In this case, the total mass on the interval $[x_0, x_1]$ changes only through the flux across the two endpoints. The corresponding flux consists of two contributions: a nonlinear transport contribution $2u^3$ arising from the nonlinear term of the equation and a dispersive contribution u_{xx} arising from the third-order dispersive operator.

For general conformable orders α and the β_i's satisfying (C1), we see that

$$\begin{aligned} P_1 =& x^{\beta_0-1} u, \\ Q_1 =& t^{\alpha-1}\Big\{2u^3 + x^{\beta_0-\beta}[(\beta_0-\beta_3)(\beta_0-\beta_2-\beta_3)u - (\beta_0+\beta_1-\beta_3-1)xu_x + x^2 u_{xx}]\Big\}. \end{aligned}$$

When $\beta_2 + 2\beta_3 - 2\beta_0 \in (0, 1]$ and $\beta_0 + \beta_1 - \beta_3 \in (0, 1]$, we can rewrite Q_1 using conformable derivatives:

$$Q_1 = t^{\alpha-1}\Big\{2u^3 + (\beta_0-\beta_3)(\beta_0-\beta_2-\beta_3) x^{\beta_0-\beta} u + \frac{\partial^{\beta_2+2\beta_3-2\beta_0}}{\partial x} \frac{\partial^{\beta_0+\beta_1-\beta_3}}{\partial x} u]\Big\}.$$

Hence, the same conservation principle remains valid, but the density and flux are modified by the conformable structure. First, the density $P_1 = x^{\beta_0-1}u$ contains the spatial weight x^{β_0-1}, indicating that the contribution of u to the total conserved quantity depends on spatial position. Second, the flux is multiplied by the temporal factor $t^{\alpha-1}$, so that the transport rate depends explicitly on time. Third, when the β_i's satisfy certain conditions, the classical dispersive contribution u_{xx} is replaced by a conformable higher-order dispersive operator, reflecting the influence of the conformable spatial structure on the propagation of the quantity represented by u. Moreover, when $\beta_0 = \beta_1 + \beta_2 + \beta_3$, the flux contains the additional term

$$(\beta_0-\beta_3)(\beta_0-\beta_2-\beta_3) x^{\beta_0-\beta} u,$$

which has no counterpart in the classical integer-order case. This term may be viewed as an additional transport mechanism generated purely by the conformable structure. Consequently, the conformable fractionalisation modifies not only the density of the conserved quantity but also the way in which it is transported through space.

The conservation laws (P_2, Q_2), (P_3, Q_3), and (P_4, Q_4) admit similar interpretations, corresponding respectively to conformable analogues of momentum, energy, and Galilean conservation laws.

3. Proof of main result

The proof follows the general framework of [47, 48]. Since a conservation law of (1.2) is defined to be a divergence expression which vanishes for all smooth solutions of (1.2), we can focus on the smooth solutions of (1.2) when we study its conservation law. By Lemma 2.2, a smooth function $u(x,t)$ is a solution of (1.2) if and only if it satisfies

$$\begin{aligned} &t^{1-\alpha}\frac{\partial u}{\partial t}+\Big[(1-\beta_1)(1-\beta_1-\beta_2)x^{1-\beta}+6x^{1-\beta_0}u^2\Big]\frac{\partial u}{\partial x}\\ &+(3-2\beta_1-\beta_2)x^{2-\beta}\frac{\partial^2 u}{\partial x^2}+x^{3-\beta}\frac{\partial^3 u}{\partial x^3}=0. \end{aligned} \tag{3.1}$$

Hence, a divergence expression $\mathrm{D}_t P+\mathrm{D}_x Q$ is a conservation law of (1.2) if and only if it is that of (3.1).

We write u_k and $u_{k,t}$ for the variables in the jet space corresponding to the partial derivative $\frac{\partial^k u}{\partial x^k}$ and $\frac{\partial^{k+1}u}{\partial x^k\partial t}$, respectively. Sometimes we also write u_0 for u and $u_{0,t}$ for the variable in the jet space corresponding to the partial derivative $\frac{\partial u}{\partial t}$. The corresponding algebraic equation of (3.1) in the jet space $M^{(n)}$ $(n\geq 3)$ is given by

$$\begin{aligned} \Delta :=&t^{1-\alpha}u_t+\big[(1-\beta_1)(1-\beta_1-\beta_2)x^{1-\beta}+6x^{1-\beta_0}u^2\big]u_x\\ &+(3-2\beta_1-\beta_2)x^{2-\beta}u_{xx}+x^{3-\beta}u_{xxx}=0, \end{aligned} \tag{3.2}$$

where $\beta=\beta_1+\beta_2+\beta_3$ is defined in (1.3). That is,

$$\begin{aligned} u_t=&-t^{\alpha-1}\Big\{\big[(1-\beta_1)(1-\beta_1-\beta_2)x^{1-\beta}+6x^{1-\beta_0}u^2\big]u_x\\ &+(3-2\beta_1-\beta_2)x^{2-\beta}u_{xx}+x^{3-\beta}u_{xxx}\Big\}. \end{aligned} \tag{3.3}$$

It is easy to see that the Fréchet derivative D_Δ of Δ is given by

$$\begin{aligned} \mathrm{D}_\Delta=&x^{3-\beta}\,\mathrm{D}_x^3+(3-2\beta_1-\beta_2)x^{2-\beta}\,\mathrm{D}_x^2+t^{1-\alpha}\,\mathrm{D}_t\\ &+\Big[6x^{1-\beta_0}u^2+(1-\beta_1)(1-\beta_1-\beta_2)x^{1-\beta}\Big]\mathrm{D}_x+12uu_xx^{1-\beta_0}. \end{aligned} \tag{3.4}$$

Then, we calculate that the adjoint operator D_Δ^* of D_Δ is given by

$$\begin{aligned} \mathrm{D}_\Delta^*=&-x^{3-\beta}\,\mathrm{D}_x^3-(6-\beta_1-2\beta_2-3\beta_3)x^{2-\beta}\,\mathrm{D}_x^2-t^{1-\alpha}\,\mathrm{D}_t-6x^{1-\beta_0}u^2\,\mathrm{D}_x\\ &-\left[7-6\beta_2-9\beta_3+\beta_1(\beta_2+2\beta_3-3)+(\beta_2+\beta_3)(\beta_2+3\beta_3)\right]x^{1-\beta}\,\mathrm{D}_x\\ &-(1-\alpha)t^{-\alpha}-6(1-\beta_0)x^{-\beta_0}u^2-(1-\beta_3)(1-\beta_2-\beta_3)(1-\beta)x^{-\beta}. \end{aligned} \tag{3.5}$$

Moreover, since $u_{n,t}=\mathrm{D}_x^n u_t$ for any integer $n\geq 0$, we see from (3.3) that the restriction of $-t^{1-\alpha}u_{n,t}$ on the smooth solution of (3.1) can be written as follows:

$$\begin{aligned} &\sum_{i=0}^{n}\binom{n}{i}\Big[\frac{\mathrm{d}^{n-i}x^{3-\beta}}{\mathrm{d}x^{n-i}}u_{i+3}+(3-2\beta_1-\beta_2)\frac{\mathrm{d}^{n-i}x^{2-\beta}}{\mathrm{d}x^{n-i}}u_{i+2}\\ &+(1-\beta_1)(1-\beta_1-\beta_2)\frac{\mathrm{d}^{n-i}x^{1-\beta}}{\mathrm{d}x^{n-i}}u_{i+1}+6\frac{\mathrm{d}^{n-i}x^{1-\beta_0}}{\mathrm{d}x^{n-i}}\,\mathrm{D}_x^i(u^2u_1)\Big]. \end{aligned} \tag{3.6}$$

Lemma 3.1. *For an integer n, we write*

$$d(n) = \frac{1}{3}\left(n(3-\beta) - \beta_1 + \beta_2 + 3\beta_3 - 3\right).$$

If a conservation law of Eq (1.2) *has a characteristic of the form* $R(x,t,u,u_1,\ldots,u_n)$ *for some smooth function R and non-negative integer n, then we have*

$$\begin{aligned} R =& x^{d(n)}F(t)u_n + \left(\frac{1}{2}nd(n)x^{d(n)-1}F(t) + x^{d(n)+\beta/3-1}K(t)\right)u_{n-1} \\ &+ H(x,t,u,u_1,\ldots,u_{n-2}) \end{aligned}$$

for some smooth functions F, H, and K, where $d = d(n)$*. When* $n = 0$*, the formula of R above reads*

$$R = x^{d(0)}F(t)u + H(x,t);$$

when $n = 1$*, it reads*

$$R = x^{d(1)}F(t)u_1 + \left(\frac{1}{2}d(1)x^{d(1)-1}F(t) + x^{d(1)+\beta/3-1}K(t)\right)u + H(x,t).$$

Proof. We write d for $d(n)$ in the proof. Let R be the characteristic of a conservation law of Eq (1.2). Then R has an equivalent characteristic of the form $R(x,t,u,u_1,\ldots,u_n)$ for some non-negative integer n, and the restriction of $\mathrm{D}_\Delta^* R$ on any solution of (1.2) or (3.1) is zero. Since

$$\mathrm{D}_t R = \frac{\partial R}{\partial t} + \sum_{i=0}^{n}\frac{\partial R}{\partial u_i}u_{i,t}, \qquad \mathrm{D}_x R = \frac{\partial R}{\partial x} + \sum_{i=0}^{n}\frac{\partial R}{\partial u_i}u_{i+1},$$

we calculate that

$$\begin{aligned} \mathrm{D}_x^2 R =& \frac{\partial R}{\partial u_n}u_{n+2} + 2\left(\sum_{i=0}^{n-1}\frac{\partial^2 R}{\partial u_i \partial u_n}u_{i+1}u_{n+1}\right) + \left(2\frac{\partial^2 R}{\partial x \partial u_n} + \frac{\partial R}{\partial u_{n-1}}\right)u_{n+1} \\ &+ R_1(x,t,u,u_1,\ldots,u_n) \end{aligned}$$

for some smooth function R_1. Moreover, we calculate that

$$\begin{aligned} \mathrm{D}_x^3 R =& \frac{\partial R}{\partial u_n}u_{n+3} + 3\left(\sum_{i=0}^{n}\frac{\partial^2 R}{\partial u_i \partial u_n}u_{i+1}u_{n+2}\right) + \left(3\frac{\partial^2 R}{\partial x \partial u_n} + \frac{\partial R}{\partial u_{n-1}}\right)u_{n+2} \\ &+ R_2(x,t,u,u_1,\ldots,u_n,u_{n+1}) \end{aligned}$$

for some smooth function R_2.

First, we observe the terms containing u_{n+3} in the restriction of $\mathrm{D}_\Delta^* R$ on the smooth solutions of (3.1). According to the calculations above, such terms appear only in $-x^{3-\beta}\,\mathrm{D}_x^3 R$ and $-t^{1-\alpha}\,\mathrm{D}_t R$. In fact, we have

$$-x^{3-\beta}\,\mathrm{D}_x^3 R = -x^{3-\beta}\frac{\partial R}{\partial u_n}u_{n+3} + \ldots,$$

where the terms not containing u_{n+3} are omitted. On the other hand, we see from (3.6) that the restriction of $-t^{1-\alpha}\,\mathrm{D}_t R$ on the smooth solutions of (3.1) can be written as

$$x^{3-\beta}\frac{\partial R}{\partial u_n}u_{n+3}+\dots.$$

Hence, the terms containing u_{n+3} in the restriction of $\mathrm{D}^*_\Delta R$ on the smooth solutions of (3.1) cancel each other out.

Now the restriction of $\mathrm{D}^*_\Delta R$ on the smooth solutions of (3.1) becomes a polynomial on u_{n+1} and u_{n+2} with coefficients on $x, t, u, \dots, u_n$. We similarly calculate that the sum of its terms containing u_{n+2} is given by

$$\begin{aligned}\Big\{ &-3x^{3-\beta}\Big(\frac{\partial^2 R}{\partial x\partial u_n}+\sum_{i=0}^{n}\frac{\partial^2 R}{\partial u_i\partial u_n}u_{i+1}\Big)\\ &+\left[(3-\beta)n-(3+\beta_1-\beta_2-3\beta_3)\right]x^{2-\beta}\frac{\partial R}{\partial u_n}\Big\}u_{n+2}.\end{aligned}$$

We write C_{n+2} for the coefficient of u_{n+2} in the expression above, which is a polynomial on u_{n+1} of degree 1 with coefficients on $x, t, u, u_1, \dots, u_n$. The coefficient of u_{n+1} in C_{n+2} is

$$-3x^{3-\beta}\frac{\partial^2 R}{\partial u_n^2},$$

so we obtain

$$R=A(x,t,u,\dots,u_{n-1})u_n+B(x,t,u,\dots,u_{n-1}) \tag{3.7}$$

for some smooth functions A and B. When $n=0$, the formula above reads

$$R=A(x,t)u+B(x,t). \tag{3.8}$$

Suppose that $n\geq 1$. Then C_{n+2} becomes a polynomial on u_n of degree 1 with coefficients on $x, t, u, u_1, \dots, u_{n-1}$, whose coefficient of u_n is

$$-3x^{3-\beta}\frac{\partial A}{\partial u_{n-1}}.$$

Then A is independent on u_{n-1}. Suppose that $n\geq 2$. Then C_{n+2} becomes a polynomial on u_{n-1} of degree 1 with coefficients on $x, t, u, u_1, \dots, u_{n-2}$, whose coefficient of u_{n-1} is

$$-3x^{3-\beta}\frac{\partial A}{\partial u_{n-2}}.$$

Then A is independent on u_{n-2}. By induction, we conclude that A is independent on $u, u_1, \dots, u_{n-1}$, i.e., we obtain

$$A=A(x,t). \tag{3.9}$$

Now C_{n+2} becomes

$$-x^{2-\beta}\left\{3x\frac{\partial A}{\partial x}-\left[(3-\beta)n-(3+\beta_1-\beta_2-3\beta_3)\right]A\right\},$$

so we obtain

$$A = x^d F(t), \tag{3.10}$$

where

$$d = \frac{1}{3}[(3-\beta)n - (3+\beta_1-\beta_2-3\beta_3)]$$

and F is some smooth function.

When $n = 0$, we have proved the lemma, so we assume that $n \geq 1$ in what follows. Then the restriction of $\mathrm{D}^*_\Delta R$ on the smooth solutions of (3.1) becomes a polynomial on u_n and u_{n+1} with coefficients on $x, t, u, \ldots, u_{n-1}$. Moreover, since R is the sum of $x^d F(t)u_n$ and $B(x, t, u, u_1, \ldots, u_{n-1})$, $\mathrm{D}^*_\Delta R$ is the sum of $\mathrm{D}^*_\Delta(x^d F u_n)$ and $\mathrm{D}^*_\Delta B$. Under the assumption $n \geq 1$, we can apply the calculation on $\mathrm{D}^*_\Delta R$ above to $\mathrm{D}^*_\Delta B$ by replacing R by B and n by $n-1$. Thus, it remains to calculate $\mathrm{D}^*_\Delta(x^d F u_n)$. Since

$$\begin{aligned}
\mathrm{D}_t R =& x^d F' u_n + x^d F u_{n,t},\\
\mathrm{D}_x R =& d x^{d-1} F u_n + x^d F u_{n+1},\\
\mathrm{D}_x^2 R =& (d^2-d)x^{d-2} F u_n + 2d x^{d-1} F u_{n+1} + x^d F u_{n+2},\\
\mathrm{D}_x^3 R =& (d^2-d)(d-2)x^{d-3} F u_n + 3(d^2-d)x^{d-2} F u_{n+1}\\
&+ 3d x^{d-1} F u_{n+2} + x^d F u_{n+3},
\end{aligned}$$

where $d = d(n)$, we deduce from (3.6) that the restriction of $\mathrm{D}^*_\Delta(x^d F u_n)$ on the smooth solutions of (3.1) can be written as

$$-\frac{1}{2}\beta n d(n) x^{d(n)+1-\beta} F u_{n+1} + R_4(x, t, u, u_1, \ldots, u_n)$$

for some smooth function R_4. Hence, the restriction of $\mathrm{D}^*_\Delta R$ on the smooth solution of (3.1) can be regarded as a polynomial of u_{n+1} of degree 1 with coefficients of functions on $x, t, u, \ldots, u_n$, and the coefficient C_{n+1} of u_{n+1} is

$$\begin{aligned}
&-\frac{1}{2}\beta n d x^{1+d-\beta} F + (3d+\beta-3)x^{2-\beta}\frac{\partial B}{\partial u_{n-1}}\\
&-3x^{3-\beta}\left[\left(\sum_{i=0}^{n-1}\frac{\partial^2 B}{\partial u_i \partial u_{n-1}}u_{i+1}\right) + \frac{\partial^2 B}{\partial x \partial u_{n-1}}\right],
\end{aligned}$$

where $d = d(n)$. We see that C_{n+1} is a polynomial of u_n of degree 1 with coefficients of functions $x, t, u, \ldots, u_{n-1}$, and the coefficient of u_n is

$$-3x^{3-\beta}\frac{\partial^2 B}{\partial u_{n-1}^2}.$$

Thus, we obtain

$$B = G(x, t, u, u_1, \ldots, u_{n-2})u_{n-1} + H(x, t, u, u_1, \ldots, u_{n-2}) \tag{3.11}$$

for some smooth functions G and H. When $n = 1$, the formula above reads

$$B = G(x, t)u + H(x, t). \tag{3.12}$$

Suppose that $n \geq 2$. Using a similar induction as we did for A, we conclude that G is independent of $u, u_1, \ldots, u_{n-2}$, i.e.,

$$G = G(x, t). \tag{3.13}$$

Now no matter whether $n = 1$ or $n \geq 2$, the coefficient C_{n+1} of u_{n+1} in the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) becomes

$$x^{1-\beta}\Big\{-\frac{1}{2}\beta n d x^d F - 3x^2\frac{\partial G}{\partial x} + (3d+\beta-3)xG\Big\}.$$

Hence, we obtain

$$G = \frac{1}{2}ndx^{d-1}F + x^{d+\beta/3-1}K(t) \tag{3.14}$$

for some smooth function K, which completes the proof. □

Lemma 3.2. *Suppose that a conservation law of Eq* (1.2) *has a characteristic which is a smooth function $R = R(x,t,u)$. We also write*

$$\begin{aligned} R_1 =& x^{\beta_0-1}t^{\alpha-1},\\ R_2 =& x^{\beta_0-1}t^{\alpha-1}u. \end{aligned}$$

(1) *When $\beta_0 \neq \beta_1+\beta_2+\beta_3$ and $\beta_0 \neq \beta_2+\beta_3$ and $\beta_0 \neq \beta_3$, we have $R = 0$.*

(2) *When $\beta_0 = \beta_1+\beta_2+\beta_3$ or $\beta_0 = \beta_2+\beta_3$ or ($\beta_0 = \beta_3$ and $\beta_1 \neq \beta_2$), we have $R = c_1R_1$ for some $c_1 \in \mathbb{R}$.*

(3) *When $\beta_0 = \beta_3$ and $\beta_1 = \beta_2$, we have $R = c_1R_1 + c_2R_2$ for some $c_1, c_2 \in \mathbb{R}$.*

Proof. We continue the calculation in the proof of Lemma 3.1 under the assumption $n = 0$. Under the assumption, we have

$$R = x^dF(t)u + H(x,t),$$

where F and H are some smooth functions and

$$d = d(0) = \frac{1}{3}\left(-\beta_1+\beta_2+3\beta_3-3\right).$$

In particular, the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) can be regarded as a polynomial of u with coefficients of functions on x and t.

The coefficient of u^2 in the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) is given by

$$x^{-\beta_0}\left(6(\beta_0-1)H - 6x\frac{\partial H}{\partial x}\right),$$

so we obtain

$$H = x^{\beta_0-1}L(t) \tag{3.15}$$

for some smooth function L. The coefficients of u^3 and u in the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) are given by

$$\begin{gathered} 2(3\beta_0+\beta_1-\beta_2-3\beta_3)x^{-\beta_0+\frac{1}{3}(-\beta_1+\beta_2-3)+\beta_3}F,\\ \frac{x^{-\frac{4\beta_1}{3}-\frac{2\beta_2}{3}-1}}{27t^\alpha}\left(27x^\beta((\alpha-1)F - t\frac{\mathrm{d}F}{\mathrm{d}t}) + 2(\beta_1-\beta_2)(2\beta_1+\beta_2)(\beta_1+2\beta_2)Ft^\alpha\right), \end{gathered}$$

respectively, which force our discussion to fall into two cases.

Case 1: $\beta_1 \neq \beta_2$ **or** $\beta_0 \neq \beta_3$**.** In this case, we see from the coefficients of u^3 and u in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) that

$$F = 0. \tag{3.16}$$

Then the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) is a linear combination of x^{β_0-1} and $x^{\beta_0-1-\beta}$ with coefficients

$$\begin{aligned} &t^{-\alpha}\left((\alpha-1)L - t\frac{\mathrm{d}L}{\mathrm{d}t}\right), \\ &-(\beta_0-\beta_3)(\beta_0-\beta_2-\beta_3)(\beta_0-\beta)L, \end{aligned}$$

respectively. Note that $\beta_0 - 1 \neq \beta_0 - 1 - \beta$.

Case 1.1: $\beta_0 \neq \beta_3$ **and** $\beta_0 \neq \beta_2 + \beta_3$ **and** $\beta_0 \neq \beta$**.** In this case, we see from the coefficient of $x^{\beta_0-1-\beta}$ in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) that $L = 0$. Therefore, we deduce that $R = 0$.

Case 1.2: $\beta_0 = \beta_3$ **or** $\beta_0 = \beta_2 + \beta_3$ **or** $\beta_0 = \beta$**.** In this case, we see from the coefficient of x^{β_0-1} in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) that $L = c_1 t^{\alpha-1}$ for some constant c_1. Therefore, we deduce that $R = c_1 R_1$.

Case 2: $\beta_1 = \beta_2$ **and** $\beta_0 = \beta_3$**.** In this case, we see from the coefficient u in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) that

$$F = c_2 t^{\alpha-1} \tag{3.17}$$

for some constant c_2. Then the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes

$$t^{-\alpha}x^{\beta_0-1}\left((\alpha-1)L - t\frac{\mathrm{d}L}{\mathrm{d}t}\right),$$

so we obtain $L = c_1 t^{\alpha-1}$ for some constant c_1. Therefore, we deduce that $R = c_1 R_1 + c_2 R_2$. Then we finish the proof of Lemma 3.2. □

Lemma 3.3. *There does not exist a conservation law of Eq* (1.2) *whose characteristic is a smooth function* $R = R(x, t, u, u_x)$ *such that* $\partial R/\partial u_x \not\equiv 0$.

Proof. Suppose that $R = R(x, t, u, u_x)$ is the characteristic of a conservation law of Eq (1.2). We continue the calculation in the proof of Lemma 3.1 under the assumption $n = 1$. Under the assumption, we have

$$R = x^d F(t)u_x + \left(\frac{1}{2}dx^{d-1}F(t) + x^{d+\beta/3-1}K(t)\right)u + H(x,t),$$

where F, H, and K are some smooth functions and

$$d = d(1) = -\frac{2}{3}(\beta_1 - \beta_3).$$

In particular, the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) can be regarded as a polynomial of u_x and u with coefficients of functions on x and t. The coefficient of $u_x^2 u$ in the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) is

$$12x^{-\beta_0+\frac{2\beta_3}{3}-\frac{2\beta_1}{3}+1}F,$$

so we obtain $F = 0$, which contradicts the condition that $x^d F = \partial R/\partial u_x \not\equiv 0$. □

Lemma 3.4. *Suppose that a conservation law of Eq* (1.2) *has a characteristic which is a smooth function $R = R(x, t, u, u_x, u_{xx})$ such that $\partial R/\partial u_{xx} \not\equiv 0$. Then we must have*

$$\beta_0 = \beta_2 = \beta_3.$$

Moreover, we write

$$\begin{aligned}
R_1 =& x^{\beta_0-1}t^{\alpha-1},\\
R_2 =& x^{\beta_0-1}t^{\alpha-1}u,\\
R_3 =& x^{-\beta_0-1}t^{\alpha-1}\left(2x^{\beta_0+\beta_1}u^3 + (1-\beta_1)xu_x + x^2u_{xx}\right),\\
R_4 =& x^{-\beta_0-1}t^{\alpha-1}\left(2\beta_0 x^{2\beta_0}t^\alpha u^3 - \frac{1}{3}\alpha x^{3\beta_0}u + \beta_0(1-\beta_0)xt^\alpha u_x + \beta_0 x^2 t^\alpha u_{xx}\right).
\end{aligned}$$

(1) *When $\beta_0 \neq \beta_1$, we have $R = c_1R_1 + c_3R_3$ for some $c_1 \in \mathbb{R}$ and $c_3 \in \mathbb{R}\backslash\{0\}$.*

(2) *When $\beta_0 = \beta_1$, we have $R = c_1R_1 + c_2R_2 + c_3R_3 + c_4R_4$ for some $c_1, c_2 \in \mathbb{R}$ and $(c_3, c_4) \in \mathbb{R}^2\backslash\{(0,0)\}$.*

Proof. We continue the calculation in the proof of Lemma 3.1 under the assumption $n = 2$. Under the assumption, we have

$$R = x^d F(t)u_{xx} + \left(\frac{1}{2}dx^{d-1}F(t) + x^{d+\beta/3-1}K(t)\right)u_x + H(x,t,u),$$

where F, H, and K are smooth functions with $F(t) \not\equiv 0$ and

$$d = \frac{1}{3}(3 - 3\beta_1 - \beta_2 + \beta_3).$$

In particular, the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) can be regarded as a polynomial of u_{xx} and u_x with coefficients of functions on x, t, and u, and the coefficient of $u_x u_{xx}$ is

$$3x^{2-\beta_1-\beta_2-\beta_3}\left(12x^{\frac{2}{3}(\beta_2+2\beta_3)-\beta_0}uF - x\frac{\partial^2 H}{\partial u^2}\right),$$

so we obtain

$$H = 2x^{-\beta_0+\frac{2\beta_2}{3}+\frac{4\beta_3}{3}-1}u^3F + uL(x,t) + M(x,t) \tag{3.18}$$

for some smooth functions L and M. Then the restriction of $\mathrm{D}_\Delta^* R$ on the solution of (3.1) can be regarded as a polynomial of u_{xx}, u_x, and u with coefficients of functions on x and t, and the coefficient of uu_x^2 is

$$4x^{\frac{1}{3}(3-3\beta_0-3\beta_1-\beta_2+\beta_3)}\left((3\beta_0 - \beta_2 - 2\beta_3)F + 3x^{\frac{1}{3}\beta}K\right),$$

so we obtain

$$K = 0 \quad \text{and} \quad 3\beta_0 - \beta_2 - 2\beta_3 = 0, \tag{3.19}$$

since $F \not\equiv 0$. Then the coefficient of u_{xx} in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes

$$\begin{aligned}
&\frac{1}{2}t^{-\alpha}x^{-\frac{3\beta_0}{2}-2\beta_1-\beta_2+1}\Bigg\{2x^{2\beta_0+\beta_1+\frac{\beta_2}{2}}\left((\alpha-1)F - t\frac{\mathrm{d}F}{\mathrm{d}t}\right)\\
&+ t^{\alpha}\Bigg[-x^{\beta_1+\frac{\beta_2}{2}+1}\left(6x\frac{\partial L}{\partial x} + (-9\beta_0 + 2\beta_1 + \beta_2 + 6)\,L\right)\\
&-\beta_0\,(\beta_0-\beta_2)\,(\beta_0+\beta_2)\,x^{\frac{\beta_0}{2}}F(t)\Bigg]\Bigg\},
\end{aligned}$$

so we obtain

$$\begin{aligned}
L =& \frac{1}{2}x^{\frac{1}{2}(\beta_0-2\beta_1-\beta_2-2)}\Bigg\{2x^{\frac{1}{3}(3\beta_0+2\beta_1+\beta_2)}P(t) + \frac{t^{-\alpha}}{3\beta_0+2\beta_1+\beta_2}\Bigg[\\
&\beta_0(\beta_0-\beta_2)(\beta_0+\beta_2)t^{\alpha}F - 4x^{\frac{1}{2}(3\beta_0+2\beta_1+\beta_2)}\left(t\frac{\mathrm{d}F}{\mathrm{d}t} + (1-\alpha)F\right)\Bigg]\Bigg\}
\end{aligned} \tag{3.20}$$

for some smooth function P. Then the coefficient of u^2 in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes

$$x^{-\beta_0}\left(6(\beta_0-1)M - 6x\frac{\partial M}{\partial x}\right),$$

so we obtain

$$M = x^{\beta_0-1}Q(t) \tag{3.21}$$

for some smooth function Q. Then the coefficient of u^3 in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes a linear combination of x^{e_1}, x^{e_2}, and x^{e_3} with coefficients

$$\begin{aligned}
&-\frac{2(3\beta_0-2\beta_1-\beta_2)t^{-\alpha}}{3\beta_0+2\beta_1+\beta_2}\left((\alpha-1)F - t\frac{\mathrm{d}F}{\mathrm{d}t}\right),\\
&\frac{(\beta_0-\beta_2)(\beta_0+\beta_2)(\beta_0+2\beta_1+\beta_2)(9\beta_0+2\beta_1+\beta_2)F}{4(3\beta_0+2\beta_1+\beta_2)},\\
&(-3\beta_0+2\beta_1+\beta_2)P,
\end{aligned}$$

respectively, where

$$e_1 = \beta_0 - 1, \quad e_2 = \frac{1}{2}\left(-\beta_0-2\beta_1-\beta_2-2\right), \quad e_3 = \frac{1}{6}\left(3\beta_0-2\beta_1-\beta_2-6\right).$$

Note that $e_i \neq e_j$ when $i \neq j$. We see from the coefficient of x^{e_2} that

$$\beta_0 - \beta_2 = 0, \tag{3.22}$$

since $F \not\equiv 0$. Then we obtain $\beta_0 = \beta_2 = \beta_3$ from (3.19) and (3.22). Then the coefficients of x^{e_1} and x^{e_3} become

$$-\frac{2(\beta_0-\beta_1)t^{-\alpha}}{2\beta_0+\beta_1}\left((\alpha-1)F - t\frac{\mathrm{d}F}{\mathrm{d}t}\right) \quad \text{and} \quad -2(\beta_0-\beta_1)P,$$

respectively, which force our discussion to fall into two cases.

Case 1: $\beta_0 - \beta_1 \neq 0$**.** In this case, we see from the coefficient of u^3 that

$$F = c_3 t^{\alpha-1} \quad \text{and} \quad P = 0 \tag{3.23}$$

for some non-zero constant c_3. Then the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes

$$t^{-\alpha} x^{\beta_0 - 1} \left((\alpha - 1)Q - t \frac{\mathrm{d}Q}{\mathrm{d}t} \right),$$

so we obtain $Q = c_1 t^{\alpha-1}$ for some constant c_1. Therefore, we deduce that $R = c_1 R_1 + c_3 R_3$.

Case 2: $\beta_0 - \beta_1 = 0$**.** In this case, we have $\beta_0 = \beta_1 = \beta_2 = \beta_3$. The coefficient of u in the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes a linear combination of x^{e_1} and x^{e_4} with coefficients

$$t^{-\alpha} \left((\alpha - 1)P - t \frac{\mathrm{d}P}{\mathrm{d}t} \right),$$
$$\frac{t^{-2\alpha}}{3\beta_0} \left(t^2 \frac{\mathrm{d}^2 F}{\mathrm{d}t^2} + (\alpha - 1) \left((2\alpha - 1)F(t) - 3t \frac{\mathrm{d}F}{\mathrm{d}t} \right) \right),$$

respectively, where e_1 is defined above and $e_4 = 2\beta_0 - 1$. Note that $e_1 \neq e_4$. Hence, we obtain

$$P = c_2 t^{\alpha-1}, \quad F = c_3 t^{\alpha-1} + c_4 t^{2\alpha-1} \tag{3.24}$$

for some constants c_2, c_3, and c_4. Then the restriction of $\mathrm{D}^*_\Delta R$ on the solution of (3.1) becomes

$$t^{-\alpha} x^{\beta_0 - 1} \left((\alpha - 1)Q - t \frac{\mathrm{d}Q}{\mathrm{d}t} \right),$$

so we obtain $Q = c_1 t^{\alpha-1}$ for some constant c_1. Therefore, we deduce that $R = c_1 R_1 + c_2 R_2 + c_3 R_3 + c'_4 R_4$ with $c'_4 = c_4/\beta_0$. Then we complete the proof. □

Lemma 3.5. *There does not exist a conservation law of Eq* (1.2) *whose characteristic is a smooth function* $R = R(x, t, u, u_1, u_2, u_3)$ *such that* $\partial R/\partial u_3 \not\equiv 0$.

Proof. We continue the calculation in the proof of Lemma 3.1 under the assumption $n = 3$. Under the assumption, we have

$$R = x^d F(t) u_{xxx} + \left[x^{d+\beta/3-1} K(t) + \frac{3}{2} d x^{d-1} F(t) \right] u_{xx} + H(x, t, u, u_x),$$

where F, K, and H are smooth functions and

$$d = d(3) = \frac{2}{3} (3 - 2\beta_1 - \beta_2).$$

The restriction of $\mathrm{D}^*_\Delta R$ on the smooth solutions of (3.1) can be written as a polynomial on u_{xx} and u_{xxx} with coefficients of functions on x, t, u, and u_x, whose coefficient of $u_{xx} u_{xxx}$ is

$$-3x^{3-\beta} \frac{\partial^2 H}{\partial u_x^2},$$

so we obtain

$$H = L(x,t,u)u_x + M(x,t,u) \tag{3.25}$$

for some smooth functions L and M. Then the restriction of $\mathrm{D}_\Delta^* R$ on the smooth solutions of (3.1) can be written as a polynomial on u_x, u_{xx}, and u_{xxx} with coefficients of functions on x, t, and u, whose coefficient of u_{xx}^2 is

$$36x^{-\beta_0-\frac{4\beta_1}{3}-\frac{2\beta_2}{3}+3}u - 3x^{3-\beta}\frac{\partial L}{\partial u},$$

so we obtain

$$L = 6x^{-\beta_0+\frac{1}{3}(\beta_2-\beta_1)+\beta_3}Fu + P(x,t) \tag{3.26}$$

for some smooth function P. Then the coefficient of $u_x u_{xxx}$ in the restriction of $\mathrm{D}_\Delta^* R$ on the smooth solutions of (3.1) becomes

$$12x^{-\beta_0-\frac{4\beta_1}{3}-\frac{2\beta_2}{3}+3}Fu,$$

so we obtain $F = 0$, which contradicts the condition that $x^d F = \partial R/\partial u_{xxx} \not\equiv 0$. □

Now by combining Lemmas 3.2–3.5, we see that the linear combination of R_1, R_2 defined in Lemma 3.2 and R_3, R_4 defined in Lemma 3.4 forms all possible characteristics of Eq (1.2) with degree no higher than 3. Their corresponding conservation laws can be easily computed, and hence we complete the proof of Theorem 1.1.

4. Conclusions

In this paper, we have studied the conservation laws of the conformable fractional mKdV equation (1.2), in which the temporal and spatial derivatives are replaced by conformable derivatives of orders α and $\beta_0, \beta_1, \beta_2, \beta_3$, respectively. Our main result, Theorem 1.1, provides a complete classification of all conservation laws of Eq (1.2) whose characteristics have order at most 3.

The four most important conservation laws of the classical mKdV equation, i.e., the mass, momentum, energy, and Galilean conservation laws, each admit a conformable counterpart, given explicitly by the pairs (P_1, Q_1), (P_2, Q_2), (P_3, Q_3), and (P_4, Q_4) in Theorem 1.1. The necessary and sufficient conditions for each of these conservation laws to hold are conditions (C1), (C2), (C3), and (C4), respectively. Furthermore, we have shown that under the appropriate conditions, these conformable conservation laws continue to provide a complete classification in the same sense as in the classical integer-order case.

Several directions remain open for future investigation. First, the present work focuses on conservation laws whose characteristics have order at most 3. It would be natural to ask under what conditions on the fractional orders that the conformable mKdV equation possesses infinitely many independent conservation laws, as in the classical case. Second, it would be of interest to extend the analysis to other conformable fractional nonlinear dispersive equations to determine whether similar structural persistence phenomena hold more broadly. Third, from a numerical standpoint, the explicitly identified conformable conservation laws could serve as the basis for structure-preserving numerical schemes for Eq (1.2), in the spirit of conservative finite difference or finite element methods for the classical mKdV equation. We leave these questions for future work.

Author contributions

Nan Wangyu: Computation, writing – original draft; Qiao Xin: Critical direction, formal analysis, conceptualization, project administration and overall strategic guidance; Yang Jiang: Computation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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