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# 指数权的加权 Lebesgue 空间中拟齐次核最优 Hilbert 型积分不等式的参数条件及应用

赵 茜[1]，洪 勇[1]，孔荫莹[2]

（1. 广州华商学院 人工智能学院，广州 511300；2. 广东财经大学 统计与数学学院，广州 510320）

**摘要：** 首先，引入拟齐次核的概念，讨论权函数为指数函数的加权 Lebesgue 空间中具有拟齐次核的 Hilbert 型积分不等式；其次，利用权系数方法及若干分析技巧，给出最优 Hilbert 型积分不等式的等价参数条件，并获得了最佳常数因子的计算公式；最后，讨论其在算子理论中的应用.

**关键词：** 加权 Lebesgue 空间；拟齐次核；Hilbert 型积分不等式；最佳搭配参数；等价条件；有界算子；算子范数

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# Parameter Conditions of Optimal Hilbert-Type Integral Inequalities with Quasi-homogeneous Kernel in Weighted Lebesgue Spaces with Exponential Weight and Applications

ZHAO Qian[1], HONG Yong[1], KONG Yinying[2]

(1. *College of Artificial Intelligence, Guangzhou Huashang College, Guangzhou* 511300, *China*;
2. *College of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou* 510320, *China*)

**Abstract:** Firstly, the concept of quasi-homogeneous kernel was introduced to discuss Hilbert-type integral inequalities with quasi-homogeneous kernel in weighted Lebesgue spaces with exponential functions. Secondly, by using the weight coefficient method and several analysis techniques, equivalent parameter condition for optimal Hilbert-type integral inequalities was given, and the calculation formula for the best constant factor was obtained. Finally, its applications in operator theory were discussed.

**Keywords:** weighted Lebesgue space; quasi-homogeneous kernel; Hilbert-type integral inequality; optimal matching parameter; equivalent condition; bounded operator; operator norm

## 0 引 言

设 $\frac{1}{p}+\frac{1}{q}=1(p>1)$，$f(x)\in L_p(0,+\infty)$，$g(y)\in L_q(0,+\infty)$，则 Hilbert 积分不等式[1]为

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**第一作者简介：** 赵 茜(1995—)，女，汉族，硕士，讲师，从事大数据分析及实分析的研究，E-mail：eunicezhao_777@163.com.

**通信作者简介：** 洪 勇(1959—)，男，汉族，硕士，教授，从事调和分析及解析不等式的研究，E-mail：hongyonggdcc@yeah.net.

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$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant\frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q,\tag{1}$$

其中常数因子$\frac{\pi}{\sin(\pi/p)}$是最佳值. 定义积分算子

$$T(f)(y)=\int_0^{+\infty}\frac{1}{x+y}f(x)\mathrm{d}x,\qquad f(x)\in L_p(0,+\infty),$$

则式(1)可等价地表示为如下算子 $T$ 的不等式:

$$\|T(f)\|_p\leqslant\frac{\pi}{\sin(\pi/p)}\|f\|_p.$$

因此 $T$ 是$L_p(0,+\infty)$中的有界算子,且算子范数$\|T\|=\frac{\pi}{\sin(\pi/p)}$.

设 $\varphi(x)>0$,引入加权 Lebesgue 空间:

$$L_p^{\varphi(x)}(a,b)=\left\{f(x):\|f\|_{p,\varphi(x)}=\left(\int_a^b\varphi(x)|f(x)|^p\mathrm{d}x\right)^{1/p}<+\infty\right\}.$$

一般地,若$\frac{1}{p}+\frac{1}{q}=1(p>1)$, $\varphi_1(x)>0$, $\varphi_2(y)>0$,则以 $K(x,y)$为核的不等式

$$\int_a^b\int_a^b K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)}\tag{2}$$

称为 Hilbert 型积分不等式,若其常数因子 $M$ 为最佳值,则式(2)称为最优不等式.

目前,关于 Hilbert 型不等式的研究已有很多成果[2-24]. 例如:文献[2]考虑指数权函数的加权 Lebesgue 空间,得到了一个具有最佳常数因子的 Hilbert 型不等式:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{(1+\mathrm{e}^x\mathrm{e}^y)^\lambda}\mathrm{d}x\mathrm{d}y\leqslant\mathrm{B}\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\|f\|_{2,\mathrm{e}^{-\lambda x}}\|g\|_{2,\mathrm{e}^{-\lambda y}},$$

其中 $\lambda>0$, $\mathrm{B}(u,v)$是 Beta 函数;文献[3]在文献[2]的基础上引入独立参数 $\sigma$ 和 $\sigma_1$,讨论了核为 $H(\mathrm{e}^{ax}\mathrm{e}^{by})$的 Hilbert 型积分不等式:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}H(\mathrm{e}^{ax}\mathrm{e}^{by})f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M\|f\|_{p,\exp\{\sigma up x\}}\|g\|_{q,\exp\{\sigma_1 bqy\}},$$

并得到了最佳常数因子的等价条件. 本文引入一类广义齐次核:

$$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y}),$$

其中 $G(u,v)$是 $\lambda$ 阶齐次非负函数,讨论指数函数加权 Lebesgue 空间中最优 Hilbert 型不等式的等价参数条件.

## 1 引 理

**引理 1** 设 $G(u,v)$是 $\lambda$ 阶齐次非负函数, $\lambda_1\lambda_2>0$, $K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})$,则

$$K(x,y)=\mathrm{e}^{\lambda\lambda_1 x}K\left(0,y-\frac{\lambda_1}{\lambda_2}x\right)=\mathrm{e}^{\lambda\lambda_2 y}K\left(x-\frac{\lambda_2}{\lambda_1}y,0\right).$$

特别地,有

$$K(0,t)=\mathrm{e}^{\lambda\lambda_2 t}K\left(-\frac{\lambda_2}{\lambda_1}t,0\right),\qquad K(t,0)=\mathrm{e}^{\lambda\lambda_1 t}K\left(0,-\frac{\lambda_1}{\lambda_2}t\right).$$

证明:根据 $G(u,v)$是 $\lambda$ 阶齐次函数,有

$$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})=\mathrm{e}^{\lambda\lambda_1 x}G(1,\mathrm{e}^{\lambda_2 y-\lambda_1 x})\mathrm{e}^{\lambda\lambda_1 x}G(\mathrm{e}^{0\lambda_1},\mathrm{e}^{\lambda_2[y-(\lambda_1/\lambda_2)x]})=\mathrm{e}^{\lambda\lambda_1 x}K\left(0,y-\frac{\lambda_1}{\lambda_2}x\right).$$

同理可证 $K(x,y)=\mathrm{e}^{\lambda\lambda_2 y}K\left(x-\frac{\lambda_2}{\lambda_1}y,0\right)$.

**引理 2** 设$\frac{1}{p}+\frac{1}{q}=1(p>1)$, $\lambda_1\lambda_2>0$, $a,b\in\mathbb{R}$, $G(u,v)$是 $\lambda$ 阶齐次非负函数, $K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})$,记

$$W_1(s)=\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{st}\,\mathrm{d}t,\qquad W_2(s)=\int_{-\infty}^{+\infty}K(t,0)\mathrm{e}^{st}\,\mathrm{d}t,$$

则有

$$\omega_1(b,p,x)=\int_{-\infty}^{+\infty}K(x,y)\mathrm{e}^{-bpy}\,\mathrm{d}y=\mathrm{e}^{\lambda_1[\lambda-(1/\lambda_2)bp]x}W_1(-bp),$$

$$\omega_2(a,q,y)=\int_{-\infty}^{+\infty}K(x,y)\mathrm{e}^{-aqx}\,\mathrm{d}x=\mathrm{e}^{\lambda_2[\lambda-(1/\lambda_1)aq]y}W_2(-aq).$$

若$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$，则$\frac{1}{\lambda_1}W_1(-bp)=\frac{1}{\lambda_2}W_2(-aq)$.

证明：根据引理 1，有

$$\omega_1(b,p,x)=\mathrm{e}^{\lambda\lambda_1 x}\int_{-\infty}^{+\infty}K\left(0,y-\frac{\lambda_1}{\lambda_2}x\right)\mathrm{e}^{-bpy}\,\mathrm{d}y=\mathrm{e}^{\lambda\lambda_1 x}\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bp[t+(\lambda_1/\lambda_2)x]}\,\mathrm{d}t=$$

$$\mathrm{e}^{\lambda_1[\lambda-(1/\lambda_2)bp]x}\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t=\mathrm{e}^{\lambda_1[\lambda-(1/\lambda_2)bp]x}W_1(-bp).$$

类似可证 $\omega_2(a,q,y)=\mathrm{e}^{\lambda_2[\lambda-(1/\lambda_1)aq]y}W_2(-aq)$.

若$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$，则

$$W_1(-bp)=\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t=\int_{-\infty}^{+\infty}K\left(-\frac{\lambda_2}{\lambda_1}t,0\right)\mathrm{e}^{(\lambda\lambda_2-bp)t}\,\mathrm{d}t=$$

$$\frac{\lambda_1}{\lambda_2}\int_{-\infty}^{+\infty}K(u,0)\mathrm{e}^{[-\lambda\lambda_1+(\lambda_1/\lambda_2)bp]u}\,\mathrm{d}u=$$

$$\frac{\lambda_1}{\lambda_2}\int_{-\infty}^{+\infty}K(u,0)\mathrm{e}^{-aqu}\,\mathrm{d}u=\frac{\lambda_1}{\lambda_2}W_2(-aq),$$

故$\frac{1}{\lambda_1}W_1(-bp)=\frac{1}{\lambda_2}W_2(-aq)$.

**引理 3**　设$\frac{1}{p}+\frac{1}{q}=1(p>1)$，$K(x,y)\geqslant 0$，$\varphi_1(x)>0$，$\varphi_2(y)>0$，$\Omega\subseteq\mathbb{R}$是一个可测集，积分算子 $T$ 为

$$T(f)(y)=\int_\Omega K(x,y)f(x)\mathrm{d}x,$$

则 Hilbert 型不等式

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)}\tag{3}$$

等价于算子不等式

$$\|T(f)\|_{p,\varphi_2^{1-p}(y)}\leqslant M\|f\|_{p,\varphi_1(x)}.\tag{4}$$

证明：若式(4)成立，则根据 Hölder 不等式，有

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y=\int_\Omega g(y)T(f)(y)\mathrm{d}y=$$

$$\int_\Omega(\varphi_2^{1/q}(y)g(y))\,(\varphi_2^{-1/q}(y)T(f)(y))\,\mathrm{d}y\leqslant$$

$$\left(\int_\Omega\varphi_2(y)|g(y)|^q\mathrm{d}y\right)^{1/q}\left(\int_\Omega\varphi_2^{-p/q}(y)|T(f)(y)|^p\mathrm{d}y\right)^{1/p}=$$

$$\|g\|_{q,\varphi_2(y)}\left(\int_\Omega\varphi_2^{1-p}(y)|T(f)(y)|^p\mathrm{d}y\right)^{1/p}=$$

$$\|g\|_{q,\varphi_2(y)}\|T(f)\|_{p,\varphi_2^{1-p}(y)}\leqslant M\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)},$$

故式(4)成立.

反之，若式(3)成立，则

$$\|T(f)\|_{p,\varphi_2^{1-p}(y)}^p=\int_\Omega\varphi_2^{1-p}(y)|T(f)(y)|^p\mathrm{d}y=$$

$$\int_\Omega \varphi_2^{1-p}(y)\left|\int_\Omega K(x,y)f(x)\mathrm{d}x\right|^p\mathrm{d}y\leqslant$$

$$\int_\Omega \varphi_2^{1-p}(y)\left|\int_\Omega K(x,y)f(x)\mathrm{d}x\right|^{p-1}\left(\int_\Omega K(x,y)|f(x)|\mathrm{d}x\right)\mathrm{d}y.$$

令 $g(y)=\varphi_2^{1-p}(y)\left|\int_\Omega K(x,y)f(x)\mathrm{d}x\right|^{p-1}$，则

$$\|T(f)\|^p_{p,\varphi_2^{1-p}(y)}\leqslant\int_\Omega g(y)\int_\Omega K(x,y)|f(x)|\mathrm{d}x\mathrm{d}y=$$

$$\int_\Omega\int_\Omega K(x,y)|f(x)|g(y)\mathrm{d}x\mathrm{d}y\leqslant$$

$$M\||f|\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)}=M\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)}=$$

$$M\|f\|_{p,\varphi_1(x)}\left(\int_\Omega\varphi_2(y)\left(\varphi_2^{1-p}(y)\left|\int_\Omega K(x,y)f(x)\mathrm{d}x\right|^{p-1}\right)^q\mathrm{d}y\right)^{1/q}=$$

$$M\|f\|_{p,\varphi_1(x)}\left(\int_\Omega\varphi_2^{1-p}(y)\left|\int_\Omega K(x,y)f(x)\mathrm{d}x\right|^p\mathrm{d}y\right)^{1/q}=$$

$$M\|f\|_{p,\varphi_1(x)}\left(\int_\Omega\varphi_2^{1-p}(y)|T(f)(y)|^p\mathrm{d}y\right)^{1/q}=$$

$$M\|f\|_{p,\varphi_1(x)}\|T(f)\|^{p-1}_{p,\varphi_2^{1-p}(y)},$$

从而可得式(4).

## 2 主要结果

**定理 1** 设$\frac{1}{p}+\frac{1}{q}=1(p>1)$，$a,b\in\mathbb{R}$，$\lambda_1\lambda_2>0$，$G(u,v)$是$\lambda$阶齐次非负函数，$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})$，则下列结论成立：

1) 记 $\varphi_1(x)=\exp\left\{\lambda_1\left[p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)+\lambda\right]x\right\}$，$\varphi_2(y)=\exp\left\{\lambda_2\left[q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)+\lambda\right]y\right\}$，则

$$A(K,f,g)=:\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant$$

$$W_1^{1/p}(-bp)W_2^{1/q}(-aq)\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)},\tag{5}$$

其中 $f(x)\in L_p^{\varphi_1(x)}(-\infty,+\infty)$，$g(y)\in L_q^{\varphi_2(y)}(-\infty,+\infty)$.

2) 若 $W_1(-bp)<+\infty$，$W_2(-aq)<+\infty$，则 $W_1^{1/p}(-bp)W_2^{1/q}(-aq)$是式(5)最佳常数因子的充分必要条件是$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$，且当$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$时，式(5)化为

$$A(K,f,g)\leqslant\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)\|f\|_{p,\exp\{apqx\}}\|g\|_{q,\exp\{bpqy\}},\tag{6}$$

其中$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)$是最佳常数因子.

证明：1) 选取 $a,b$ 为搭配参数，利用 Hölder 不等式及引理 2，有

$$A(K,f,g)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\mathrm{e}^{ax-by}f(x))(\mathrm{e}^{by-ax}g(y))K(x,y)\mathrm{d}x\mathrm{d}y\leqslant$$

$$\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{(ax-by)p}|f(x)|^pK(x,y)\mathrm{d}x\mathrm{d}y\right)^{1/p}\times$$

$$\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{(by-ax)q}|g(y)|^qK(x,y)\mathrm{d}x\mathrm{d}y\right)^{1/q}=$$

$$\left(\int_{-\infty}^{+\infty}\mathrm{e}^{apx}|f(x)|^p\omega_1(b,p,x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{+\infty}\mathrm{e}^{bqy}|g(y)|^q\omega_2(a,q,y)\mathrm{d}y\right)^{1/q}=$$

$$W_1^{1/p}(-bp)W_2^{1/q}(-aq)\left(\int_{-\infty}^{+\infty}\exp\left\{\lambda_1\left[p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)+\lambda\right]x\right\}|f(x)|^p\mathrm{d}x\right)^{1/p}\times$$

$$\left(\int_{-\infty}^{+\infty}\exp\left\{\lambda_2\left[q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)+\lambda\right]y\right\}|g(y)|^q\mathrm{d}x\right)^{1/q}=$$
$$W_1^{1/p}(-bp)W_2^{1/q}(-aq)\|f\|_{p,\varphi_1(x)}\|g\|_{q,\varphi_2(y)}.$$

故式(5)成立.

2) 充分性. 设$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$，则计算可得 $\varphi_1(x)=\exp\{apqx\}$，$\varphi_2(y)=\exp\{bpqy\}$. 又根据引理 2，有$\frac{1}{\lambda_1}W_1(-bp)=\frac{1}{\lambda_2}W_2(-aq)$，于是式(5)可化为式(6). 从而只需证明式(6)的常数因子最佳即可.

若式(6)的常数因子不是最佳的，则存在常数 $M_0<\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)$，使得

$$A(K,f,g)\leqslant M_0\|f\|_{p,\exp\{apqx\}}\|g\|_{q,\exp\{bpqy\}}.$$

取充分小的 $\varepsilon>0$ 及足够大的正数 $N$，令

$$f(x)=\begin{cases}\exp\{(-apq-|\lambda_1|\varepsilon)x/p\}, & x\geqslant N,\\ 0, & x<N,\end{cases}$$
$$g(y)=\begin{cases}\exp\{(-bpq-|\lambda_2|\varepsilon)y/q\}, & y\geqslant 0,\\ 0, & y<0,\end{cases}$$

则有

$$\|f\|_{p,\exp\{apqx\}}\|g\|_{q,\exp\{bpqy\}}=\left(\int_N^{+\infty}\mathrm{e}^{-|\lambda_1|\varepsilon x}\mathrm{d}x\right)^{1/p}\left(\int_0^{+\infty}\mathrm{e}^{-|\lambda_2|\varepsilon y}\mathrm{d}y\right)^{1/q}=$$
$$\left(\frac{1}{|\lambda_1|\varepsilon}\mathrm{e}^{-|\lambda_1|\varepsilon N}\right)^{1/p}\left(\frac{1}{|\lambda_2|\varepsilon}\right)^{1/q}=$$
$$\frac{1}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\mathrm{e}^{-|\lambda_1|\varepsilon N/p}.$$

根据引理 1，有

$$A(K,f,g)=\int_N^{+\infty}\exp\left\{\left(-aq-\frac{|\lambda_1|\varepsilon}{p}\right)x\right\}\left(\int_0^{+\infty}K(x,y)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)y\right\}\mathrm{d}y\right)\mathrm{d}x=$$
$$\int_N^{+\infty}\exp\left\{\left(-aq-\frac{|\lambda_1|\varepsilon}{p}\right)x\right\}\times$$
$$\left(\int_0^{+\infty}K\left(0,y-\frac{\lambda_1}{\lambda_2}x\right)\mathrm{e}^{\lambda\lambda_1 x}\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)y\right\}\mathrm{d}y\right)\mathrm{d}x=$$
$$\int_N^{+\infty}\exp\left\{\left(\lambda\lambda_1-aq-\frac{|\lambda_1|\varepsilon}{p}\right)x\right\}\times$$
$$\left(\int_{-(\lambda_1/\lambda_2)x}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)\left(t+\frac{\lambda_1}{\lambda_2}x\right)\right\}\mathrm{d}t\right)\mathrm{d}x=$$
$$\int_N^{+\infty}\exp\left\{\left[\lambda\lambda_1-aq-\frac{|\lambda_1|\varepsilon}{p}-\frac{\lambda_1}{\lambda_2}\left(bp+\frac{|\lambda_1|\varepsilon}{q}\right)\right]x\right\}\times$$
$$\left(\int_{-(\lambda_1/\lambda_2)x}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)t\right\}\mathrm{d}t\right)\mathrm{d}x=$$
$$\int_N^{+\infty}\mathrm{e}^{|\lambda_1|\varepsilon x}\left(\int_{-(\lambda_1/\lambda_2)x}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)t\right\}\mathrm{d}t\right)\mathrm{d}x\geqslant$$
$$\int_N^{+\infty}\mathrm{e}^{|\lambda_1|\varepsilon x}\mathrm{d}x\int_{-(\lambda_1/\lambda_2)N}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)t\right\}\mathrm{d}t=$$
$$\frac{1}{\varepsilon|\lambda_1|}\mathrm{e}^{-|\lambda_1|\varepsilon N}\int_{-(\lambda_1/\lambda_2)N}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)t\right\}\mathrm{d}t.$$

综上可得

$$\frac{1}{|\lambda_1|}\mathrm{e}^{-|\lambda_1|\varepsilon N}\int_{-(\lambda_1/\lambda_2)N}^{+\infty}K(0,t)\exp\left\{\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)t\right\}\mathrm{d}t\leqslant M_0\frac{1}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\mathrm{e}^{-|\lambda_1|\varepsilon N/p}.$$

令 $\varepsilon\to 0^+$，利用 Fatou 引理[25]，有

$$\frac{1}{|\lambda_1|}\int_{-(\lambda_1/\lambda_2)N}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t\leqslant M_0\,\frac{1}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}},$$

再令 $N\to+\infty$，得

$$\frac{1}{|\lambda_1|}W_1(-bp)=\frac{1}{|\lambda_1|}\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t\leqslant M_0\,\frac{1}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}},$$

于是有 $\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)\leqslant M_0$，与 $M_0<\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)$ 矛盾，故 $\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)$ 是式(6)的最佳常数因子.

必要性. 设 $W_1^{1/p}(-bp)W_2^{1/q}(-aq)$ 是式(5)的最佳常数因子，记 $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\lambda=c$，$a_1=a-\frac{\lambda_1 c}{pq}$，$b_1=b-\frac{\lambda_2 c}{pq}$，则

$$\frac{1}{\lambda_1}a_1q+\frac{1}{\lambda_2}b_1p=\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\frac{c}{p}-\frac{c}{q}=\lambda+c-c=\lambda,$$

$$\varphi_1(x)=\exp\left\{\lambda_1\left[p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)+\lambda\right]x\right\}=\exp\left\{\lambda_1\left[p\left(\frac{a_1}{\lambda_1}-\frac{b_1}{\lambda_2}\right)+\lambda\right]x\right\}=\exp\{a_1pqx\},$$

$$\varphi_2(y)=\exp\left\{\lambda_2\left[q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)+\lambda\right]y\right\}=\exp\left\{\lambda_2\left[q\left(\frac{b_1}{\lambda_2}-\frac{a_1}{\lambda_1}\right)+\lambda\right]y\right\}=\exp\{b_1pqy\},$$

$$W_2(-aq)=\int_{-\infty}^{+\infty}K(t,0)\mathrm{e}^{-aqt}\,\mathrm{d}t=\int_{-\infty}^{+\infty}K\left(0,-\frac{\lambda_1}{\lambda_2}t\right)\mathrm{e}^{(\lambda\lambda_1-aq)t}\,\mathrm{d}t=$$

$$\frac{\lambda_2}{\lambda_1}\int_{-\infty}^{+\infty}K(0,u)\mathrm{e}^{(-bp+\lambda_2c)u}\,\mathrm{d}u=\frac{\lambda_2}{\lambda_1}\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{(-bp+\lambda_2c)t}\,\mathrm{d}t,$$

于是式(5)可化为

$$A(K,f,g)=W_1^{1/p}(-bp)\left(\frac{\lambda_2}{\lambda_1}\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{(-bp+\lambda_2c)t}\,\mathrm{d}t\right)^{1/q}\|f\|_{p,\exp\{a_1pqx\}}\|g\|_{q,\exp\{b_1pqy\}}.\tag{7}$$

由于式(5)的常数因子是最佳的，故式(7)的常数因子为

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1^{1/p}(-bp)\left(\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{(-bp+\lambda_2c)t}\,\mathrm{d}t\right)^{1/q},$$

因为 $\frac{1}{\lambda_1}a_1q+\frac{1}{\lambda_2}b_1p=\lambda$，故根据充分条件的证明可知，式(7)的最佳常数因子应为

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)=\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}\int_{-\infty}^{+\infty}K(0,t)\exp\left\{-bpt+\frac{\lambda_2c}{q}t\right\}\mathrm{d}t.$$

于是

$$\int_{-\infty}^{+\infty}K(0,t)\exp\left\{-bpt+\frac{\lambda_2c}{q}t\right\}\mathrm{d}t=W_1^{1/p}(-bp)\left(\int_{-\infty}^{+\infty}K(0,t)\exp\{(-bp+\lambda_2c)t\}\,\mathrm{d}t\right)^{1/q}.\tag{8}$$

利用 Hölder 不等式，有

$$\int_{-\infty}^{+\infty}K(0,t)\exp\left\{\left(-bp+\frac{\lambda_2c}{q}\right)t\right\}\mathrm{d}t=\int_{-\infty}^{+\infty}1\cdot\exp\left\{\frac{\lambda_2c}{q}t\right\}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t\leqslant$$

$$\left(\int_{-\infty}^{+\infty}1^pK(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t\right)^{1/p}\left(\int_{-\infty}^{+\infty}\mathrm{e}^{\lambda_2ct}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t\right)^{1/q}=$$

$$W_1^{1/p}(-bp)\left(\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{(-bp+\lambda_2c)t}\,\mathrm{d}t\right)^{1/q}.\tag{9}$$

由式(8)知式(9)取等号，再根据 Hölder 不等式取等号的条件，可得 $\mathrm{e}^{\lambda_2ct}=$ 常数，从而 $c=0$，即 $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$.

## 3 应 用

根据引理 3 和定理 1，可得下面关于算子理论的定理.

**定理 2**　设$\frac{1}{p}+\frac{1}{q}=1(p>1)$，$a,b\in\mathbb{R}$，$\lambda_1\lambda_2>0$，$G(u,v)$是$\lambda$阶齐次非负可测函数，$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})$，$W_1(-bp)<+\infty$，$W_2(-aq)<+\infty$，积分算子$T$为

$$T(f)(y)=\int_{-\infty}^{+\infty}K(x,y)f(x)\mathrm{d}x.$$

1）记$\varphi_1(x)=\exp\left\{\lambda_1\left[p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)+\lambda\right]x\right\}$，$\varphi_2(y)=\exp\left\{\lambda_2\left[q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)+\lambda\right]y\right\}$，则$T$是从$L_p^{\varphi_1(x)}(-\infty,+\infty)$到$L_p^{\varphi_2^{1-p}(y)}(-\infty,+\infty)$的有界算子，且$T$的算子范数

$$\|T\|\leqslant W_1^{1/p}(-bp)\,W_2^{1/q}(-aq).$$

2）当且仅当$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$时，$T$的算子范数$\|T\|=W_1^{1/p}(-bp)W_2^{1/q}(-aq)$. 当$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda$时，有$\varphi_1(x)=\exp\{apqx\}$，$\varphi_2(y)=\exp\{bpqy\}$，且

$$\|T\|=W_1^{1/p}(-bp)W_2^{1/q}(-aq)=\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp).$$

**推论 1**　设$\frac{1}{p}+\frac{1}{q}=1(p>1)$，$\lambda>0$，$\lambda_1\lambda_2>0$，$k_1>0$，$k_2>0$，则积分算子

$$T(f)(y)=\int_{-\infty}^{+\infty}\frac{f(x)}{(k_1\mathrm{e}^{\lambda_1 x}+k_2\mathrm{e}^{\lambda_2 y})^{\lambda}}\mathrm{d}x$$

是从$L_p^{\exp\{-\lambda\lambda_1 x\}}(-\infty,+\infty)$到$L_p^{\exp\{-\lambda\lambda_2(1-p)y\}}(-\infty,+\infty)$的有界算子，且$T$的算子范数为

$$\|T\|=\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}k_1^{\lambda/p}k_2^{\lambda/q}}\mathrm{B}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right),$$

其中$\mathrm{B}(s,t)$是 Beta 函数.

证明：令$G(u,v)=\frac{1}{(k_1u+k_2v)^{\lambda}}$，则$G(u,v)$是$-\lambda$阶齐次函数，且

$$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})=\frac{1}{(k_1\mathrm{e}^{\lambda_1 x}+k_2\mathrm{e}^{\lambda_2 y})^{\lambda}}.$$

取$a=-\frac{\lambda\lambda_1}{pq}$，$b=-\frac{\lambda\lambda_2}{pq}$，则

$$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=-\lambda,\quad \exp\{apqx\}=\exp\{-\lambda\lambda_1 x\},\quad \exp\{bpq(1-p)y\}=\exp\{-\lambda\lambda_2(1-p)y\},$$

且

$$\begin{aligned}W_1(-bp)&=\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\mathrm{d}t=\int_{-\infty}^{+\infty}\frac{1}{(k_1+k_2\mathrm{e}^{\lambda_2 t})^{\lambda}}\mathrm{e}^{(\lambda\lambda_2/q)t}\mathrm{d}t=\\&\frac{1}{k_1^{\lambda}}\int_{-\infty}^{+\infty}\frac{1}{[1+(k_2/k_1)\mathrm{e}^{\lambda_2 t}]^{\lambda}}\mathrm{e}^{(\lambda\lambda_2/q)t}\mathrm{d}t=\\&\frac{1}{k_1^{\lambda}}\int_{-\infty}^{+\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{k_1}{k_2}u\right)^{\lambda/q}\frac{1}{|\lambda_2|}u^{-1}\mathrm{d}u=\\&\frac{1}{|\lambda_2|}\frac{1}{k_1^{\lambda/p}k_2^{\lambda/q}}\int_{-\infty}^{+\infty}\frac{1}{(1+u)^{\lambda}}u^{\lambda/q-1}\mathrm{d}u=\\&\frac{1}{|\lambda_2|k_1^{\lambda/p}k_2^{\lambda/q}}\mathrm{B}\left(\frac{\lambda}{q},\lambda-\frac{\lambda}{q}\right)=\frac{1}{|\lambda_2|k_1^{\lambda/p}k_2^{\lambda/q}}\mathrm{B}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right).\end{aligned}$$

于是可得

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)=\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}k_1^{\lambda/p}k_2^{\lambda/q}}\mathrm{B}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right).$$

综上并根据定理 2 知结论成立.

**推论 2**　设$\frac{1}{p}+\frac{1}{q}=1(p>1)$，$\lambda_1\lambda_2>0$，$0<\sigma<\lambda$，$k_1>0$，$k_2>0$，则积分算子

$$T(f)(y)=\int_{-\infty}^{+\infty}\frac{f(x)}{(k_1+k_2\mathrm{e}^{\lambda_1 x-\lambda_2 y})^{\lambda}}\mathrm{d}x$$

是从 $L_p^{\exp\{-\lambda_1\sigma px\}}(-\infty,+\infty)$ 到 $L_p^{\exp\{-\lambda_2\sigma py\}}(-\infty,+\infty)$ 的有界算子，且 $T$ 的算子范数为

$$\|T\| = \frac{k_1^{\sigma-\lambda}k_2^{-\sigma}}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\mathrm{B}(\sigma,\lambda-\sigma).$$

证明：令 $G(u,v)=\dfrac{1}{(k_1+k_2u/v)^{\lambda}}$，则 $G(u,v)$ 是 0 阶齐次函数，且

$$K(x,y)=G(\mathrm{e}^{\lambda_1 x},\mathrm{e}^{\lambda_2 y})=\frac{1}{(k_1+k_2\mathrm{e}^{\lambda_1 x-\lambda_2 y})^{\lambda}}.$$

取 $a=-\dfrac{\lambda_1\sigma}{q}$，$b=\dfrac{\lambda_2\sigma}{p}$，则

$$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=0,\quad \exp\{apqx\}=\exp\{-\lambda_1\sigma px\},\quad \exp\{bpq(1-p)y\}=\exp\{-\lambda_2\sigma py\},$$

且

$$W_1(-bp)=\int_{-\infty}^{+\infty}K(0,t)\mathrm{e}^{-bpt}\,\mathrm{d}t=\int_{-\infty}^{+\infty}\frac{1}{(k_1+k_2\mathrm{e}^{-\lambda_2 t})^{\lambda}}\mathrm{e}^{-\lambda_2\sigma t}\,\mathrm{d}t=$$

$$\frac{1}{k_1^{\lambda}}\int_{-\infty}^{+\infty}\frac{1}{[1+(k_2/k_1)\mathrm{e}^{-\lambda_2 t}]^{\lambda}}\mathrm{e}^{-\lambda_2\sigma t}\,\mathrm{d}t=$$

$$\frac{1}{|\lambda_2|}k_1^{\sigma-\lambda}k_2^{-\sigma}\int_{-\infty}^{+\infty}\frac{1}{(1+u)^{\lambda}}u^{\sigma-1}\,\mathrm{d}u=\frac{1}{|\lambda_2|}k_1^{\sigma-\lambda}k_2^{-\sigma}\mathrm{B}(\sigma,\lambda-\sigma),$$

于是可得

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(-bp)=\frac{k_1^{\sigma-\lambda}k_2^{-\sigma}}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\mathrm{B}(\sigma,\lambda-\sigma).$$

综上并根据定理 2 知结论成立.

若在推论 2 中取 $\sigma=\dfrac{1}{p}$，$\lambda=1$，则可得：

**推论 3** 设 $\dfrac{1}{p}+\dfrac{1}{q}=1(p>1)$，$\lambda_1\lambda_2>0$，$k_1>0$，$k_2>0$，则积分算子

$$T(f)(y)=\int_{-\infty}^{+\infty}\frac{f(x)}{k_1+k_2\mathrm{e}^{\lambda_1 x-\lambda_2 y}}\mathrm{d}x$$

是从 $L_p^{\exp\{-\lambda_1 x\}}(-\infty,+\infty)$ 到 $L_p{}^{\exp\{-\lambda_2 y\}}(-\infty,+\infty)$ 的有界算子，且 $T$ 的算子范数为

$$\|T\|=\frac{k_1^{-1/q}k_2^{-1/p}}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\mathrm{B}\left(\frac{1}{p},1-\frac{1}{p}\right)=\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}k_1^{1/q}k_2^{1/p}}\frac{\pi}{\sin(\pi/p)}.$$

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