

Conformal minimal 2-spheres with constant curvature in $\mathbb{H}P^n$

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Abstract We construct new conformal minimal 2-spheres of constant curvature linearly full in the quaternion projective space $\mathbb{H}P^n$ by the twistor map $\pi: \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$ in a systematic way. That is, we construct 2-spheres linearly full in $\mathbb{C}P^{2n+1}$ or $\mathbb{C}P^{2n}$ satisfying horizontal condition explicitly. (In the latter case, we consider the natural embedding of the map). We prove all 2-spheres we constructed are non-homogeneous.

Keywords quaternionic projective space; harmonic sequence; homogeneous 2-sphere; constantly curved minimal 2-sphere

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$\mathbb{H}P^n$ 中的常曲率共形极小 2 维球面

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摘要 通过扭映射 $\pi: \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$ 构造 $\mathbb{H}P^n$ 中的线性满常曲率共形极小 2 维球面。亦即构造了在 $\mathbb{C}P^{2n+1}$ 或 $\mathbb{C}P^{2n}$ 中的水平线性满常曲率共形极小 2 维球面(对于后一种情况, 考虑其在 $\mathbb{C}P^{2n+1}$ 的自然嵌入)。进而证明这种方法构造的球面都是非齐性的。

关键词 四元数射影空间; 调和序列; 齐性球面; 常曲率极小 2 维球面

Calabi^[1] proved in 1967 that a minimal 2-sphere S^2 of constant curvature in S^{n-1} is unique up to a rigid rotation. Subsequently, considerable mathematical effort have been put into the study of rigidity of 2-spheres in a variety of ambient spaces. In 1985 Bolton et al.^[2] proved that any linearly full conformal minimal immersion of constant curvature from S^2 to $\mathbb{C}P^n$ belongs to the Veronese sequence, up to a rigid motion. In this paper, we study the construction of some conformal minimal two-spheres

with constant curvature in the quaternion projective space $\mathbb{H}P^n$.

In 1990, Ohnita^[3] obtained a series of homogeneous minimal 2-spheres in $\mathbb{H}P^n$ by the representation of Lie algebra. He conjectured that this series exhausts all proper minimal isometric immersions.

In 1991, Bahy-El-Dien and Wood^[4] developed a vector bundle technique to construct minimal 2-spheres in $\mathbb{H}P^n$, based on the important fact that

$\mathbb{HP}^n$ is a totally geodesic submanifold of $G(2, 2n; \mathbb{C})$.

This procedure proved especially useful for constructing minimal 2-spheres in various ambient spaces under additional assumptions. In 2014, He and Jiao^[5] classified linearly full minimal constantly curved ones in $\mathbb{HP}^2$. Furthermore, they classified homogeneous minimal 2-spheres in $\mathbb{HP}^n$ by the method of moving frames and harmonic sequence.

Now, it is natural to ask the following question: do there exist nonhomogeneous constantly curved conformal minimal 2-spheres immersed in $\mathbb{HP}^n$?

The twistor map is a powerful tool for studying this question. In 2017, Chen and Jiao^[6] constructed 8 families of minimal surfaces in $\mathbb{HP}^2$ using the twistor map. Gao and Jiao^[7] constructed examples of conformal minimal two-spheres with constant curvatures $4/7, 4/19, 4/27$ by the twistor map $\pi: \mathbb{CP}^7 \rightarrow \mathbb{HP}^3$. Later on, Jiao and Cui^[8] discovered some important properties of the harmonic sequence induced by a horizontal harmonic 2-sphere. Furthermore, they obtained a nonhomogeneous horizontal 2-sphere. Zhang and Jiao^[9] showed that $V_0^{2n}, n \geq 3$ can be rotated horizontally and constructed minimal 2-spheres in $\mathbb{HP}^n, n = 4, 5, 6, 7$. In this paper, we construct 2-spheres linearly full in $\mathbb{CP}^{2n+1}$ or $\mathbb{CP}^{2n}$ satisfying horizontal condition explicitly by the twistor maps.

Using the expression of the norm of first ∂ return map in Jiao et al.^[10], we prove all 2-spheres constructed by our method are non-homogeneous.

1 Geometry of quaternionic projective space

The quaternion algebra $\mathbb{H}$ is regarded as a 4-dimensional real vector space with the basis $1, i, j, k$. The multiplication of rings in $\mathbb{H}$ is defined as

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k = -ji, ik = -j = -ki, jk = -k = -kj.$$

In particular, $\mathbb{H}$ is a two-dimensional right module on $\mathbb{C}$ with the basis $1, j$. For $z = z_1 + jz_2$, $w = w_1 + jw_2 \in \mathbb{H}$, we have

$$zw = (z_1w_1 - \bar{z}_2w_2) + j(z_2w_1 + \bar{z}_1w_2).$$

There is a natural conjugation in $\mathbb{H}$, denoted by

$$(z_1 + jz_2)^* = \bar{z}_1 - jz_2.$$

Let $\mathbb{H}^n$ be the set of n -tuples of quaternions written as column vectors. We can identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$:

$$z_1 + jz_2 \leftrightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The quaternionic projective space $\mathbb{HP}^n$ is defined to be

$$\mathbb{HP}^n = \{[\mathbf{x}]_{\mathbb{H}}; \mathbf{x} \in \mathbb{H}^{n+1} - \{0\}\}$$

where $[\mathbf{x}_1]_{\mathbb{H}} = [\mathbf{x}_2]_{\mathbb{H}} \Leftrightarrow \mathbf{x}_1 = \mathbf{x}_2 \cdot h, h \in \mathbb{H} - \{0\}$.

The quaternionic inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ on $\mathbb{H}^n$ is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle_{\mathbb{H}} = (\mathbf{q}^*)^T \mathbf{p} = \sum_{k=1}^n q_k^* \cdot p_k, \forall \mathbf{p}, \mathbf{q} \in \mathbb{H}^n,$$

where $\mathbf{p} = (p_1, p_2, \dots, p_n)^T, \mathbf{q} = (q_1, q_2, \dots, q_n)^T$.

Define a conjugate linear map J on $\mathbb{C}^{2n}$

$$J: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}, J\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \mapsto \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}.$$

The Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\mathbb{C}^n$ is defined by $\langle \mathbf{Z}, \mathbf{W} \rangle_{\mathbb{C}} = \sum_{i=1}^n \bar{w}_i z_i$, where $\mathbf{Z} = (z_1, z_2, \dots, z_n)^T$ and $\mathbf{W} = (w_1, w_2, \dots, w_n)^T$. Suppose that $p_k = a_k + jb_k, q_k = c_k + jd_k, p_k, q_k \in \mathbb{H}, a_k, b_k, c_k, d_k \in \mathbb{C}$.

We obtain the following equation:

$$\langle \mathbf{p}, \mathbf{q} \rangle_{\mathbb{H}} = \langle \mathbf{p}, \mathbf{q} \rangle_{\mathbb{C}} + j \langle \mathbf{p}, J\mathbf{q} \rangle_{\mathbb{C}}.$$

The (quaternionic) symplectic group is defined as

$$\mathrm{Sp}(n, \mathbb{H}) = \{A \in \mathrm{Gl}(n; \mathbb{H}) | (A^*)^T \cdot A = I_n\}.$$

The quaternion square matrix $M_n(\mathbb{H})$ can be embedded into the complex square matrix $M_{2n}(\mathbb{C})$:

$$M_n(\mathbb{H}) \rightarrow M_{2n}(\mathbb{C}), A + jB \mapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix},$$

where $A, B \in M_n(\mathbb{C})$. Let

$$B_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $M_n(\mathbb{H})$ is a subset of $M_{2n}(\mathbb{C})$, and $M_n(\mathbb{H}) = \{N \in M_{2n}(\mathbb{C}) | \bar{N}B_n = B_n N\}$.

Recall that $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$,

where $\mathrm{Sp}(2n, \mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) : A^T B_n A = B_n\}$.

For $A \in \mathrm{Sp}(n+1), [v]_{\mathbb{H}} \in \mathbb{HP}^n$, $\mathrm{Sp}(n+1)$ acts on $\mathbb{HP}^n$ by

$$A[v]_{\mathbb{H}} = [A \cdot v]_{\mathbb{H}}. \quad (1)$$

It can be shown that this action is transitive and $G_0 = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$ is the isotropy group of this action at $[(1, 0, \dots, 0)^T]_{\mathbb{H}}$. As a result,

$$\mathbb{HP}^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n). \quad (2)$$

In other words, we prove that $\mathbb{HP}^n$ is a homogeneous manifold.

We define the twistor map π as

$$\pi: \mathbb{CP}^{2n+1} \rightarrow \mathbb{HP}^n$$

$$\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]_{\mathbb{C}} \mapsto [\alpha + j\beta]_{\mathbb{H}}, \alpha, \beta \in \mathbb{C}^{n+1}.$$

Therefore, the fiber of the point $[\alpha + j\beta]_{\mathbb{H}} \in \mathbb{HP}^n$ is the complex line spanned by $\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]_{\mathbb{C}}$ and $\left[\begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} \right]_{\mathbb{C}}$.

Suppose that $M = X + jY \in \mathrm{Sp}(n+1)$, $M = (M_0, M_1, \dots, M_n)$, $X = (X_0, X_1, \dots, X_n)$, $Y = (Y_0, Y_1, \dots, Y_n)$, therefore,

$$M_k = X_k + jY_k, 0 \leq k \leq n.$$

Then we have two projections: $\pi_1(M) = \left[\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \right]_{\mathbb{C}}$, $\pi_2(M) = [M_0]_{\mathbb{H}}$. By linear algebra, π_1 is a surjection. Therefore, $\pi_2 = \pi \circ \pi_1$.

The holomorphic curvature of $\mathbb{CP}^n$ is equal to 4 under the Fubini-Study metric. Then, it is obvious that the twistor map π is a Riemannian submersion^[11].

Suppose that we have two Riemann manifolds M , N , and a map $\pi: M \rightarrow N$, which is a submersion. The horizontal distribution $\mathcal{H}$ at $p \in M$ is

$$\mathcal{H}_p := \{v \in T_p M \mid \pi_* v \neq 0 \in T_{\pi(p)} N\}.$$

The submersion π is a Riemannian submersion if and only if π is a submersion and maps the horizontal distribution $\mathcal{H}_p$ to the tangent space of $T_p N$ isometrically. That is, $g^M(X_p, Y_p) = g^N(\pi_* X_p, \pi_* Y_p)$. In this case, the horizontal distribution in the Riemannian manifold is $\mathcal{H}_q = \{v \in T_p M \mid v \perp \mathrm{Ker}(\pi_*)|_q\}$.

Definition 1.1 Let L, M, N be Riemannian manifolds, $\pi: M \rightarrow N$ is a Riemannian submersion, and $f: L \rightarrow M$ is a smooth mapping. We say f is a horizontal map if and only if $df_x(v) \in \mathcal{H}_{f(x)}$, $\forall x \in L$, $\forall v \in T_x L$.

The following lemmas are needed.

The first lemma was proved in Eells and

Sampson^[12].

Lemma 1.1 Let $f: M \rightarrow N$ be a conformal immersion, where M and N are Riemannian manifolds, then f is minimal if and only if f is harmonic.

The next lemma was proved in Eells and Wood^[13].

Lemma 1.2 Let M, Y, N be Riemannian manifolds, $\pi: Y \rightarrow N$ a Riemannian submersion, and $f: M \rightarrow Y$ a horizontal immersion with respect to π . If f is harmonic, then $\pi \circ f$ is also harmonic.

Consequently,

Lemma 1.3 If $f: M \rightarrow \mathbb{CP}^{2n+1}$ is a horizontal conformal minimal immersion, then the sphere $\pi \circ f: M \rightarrow \mathbb{HP}^n$ is conformal minimal, where π is the twistor map and M is a Riemann surface.

2 Harmonic sequence

In this section, we will introduce the theory of harmonic sequence independently developed by Chern and Wolfson^[14], and by Burstall and Wood^[15].

Let M be a Riemann surface. Assume that ψ is a Riemannian submersion, $\psi: M \rightarrow G(k, n; \mathbb{C})$, where $G(k, n; \mathbb{C})$ is the complex Grassmannian of k -plane in the n -dimension. We identify the map ψ with the subbundle of the trivial bundle, which implies that

$$\psi_x := \{(x, v) \mid x \in M, v \in \psi(x)\} \subset M \times \mathbb{C}^n.$$

Definition 2.1 Suppose that we have two bundles $E \perp L$ which are the subbundles of the trivial bundle, we define the ∂ -second fundamental form $A'_{E,L}$ as:

$$A'_{E,L}: E \rightarrow L, (x_0, v_n) \mapsto \pi_L \frac{\partial v}{\partial z}. \quad (3)$$

Suppose that $\underline{\psi}: M^2 \rightarrow G(k, n; \mathbb{C})$ is harmonic, define the ∂ -Gauss bundle of ψ as $\underline{\psi} := \mathrm{Im} A'_{\underline{\psi}, \underline{\psi}^\perp}$.

In a similar fashion, we can define $\bar{\partial}$ -second fundamental form and $\bar{\partial}$ -Gauss bundle.

By Bahy-El-Dien and Wood^[4], we have the following result:

Proposition 2.1 Suppose that $\pi \circ f: S^2 \rightarrow \mathbb{HP}^n$ is a minimal 2-sphere constructed by projecting the horizontal 2-sphere in $\mathbb{CP}^{2n+1}$ under the twistor

map, then the map $\psi: f \oplus Jf$ is defined by the following identification:

$$\mathbb{H}P^n = \{V \in G(2, 2n+2; \mathbb{C}) | V = JV\}, \pi \circ f \leftrightarrow \psi.$$

Let $\psi: M \rightarrow G(k, n; \mathbb{C})$ be a harmonic immersion. It is well known that ψ is harmonic if and only if $A'_{\psi, \psi^\perp}$ is holomorphic. We construct two sequences generated by ψ :

$$\begin{aligned} \underline{\psi}_0 &= \underline{\psi} \xrightarrow{\partial} \underline{\psi}_1 \xrightarrow{\partial} \underline{\psi}_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{\psi}_k \xrightarrow{\partial} \cdots, \\ \underline{\psi}_0 &= \underline{\psi} \xrightarrow{\bar{\partial}} \underline{\psi}_{-1} \xrightarrow{\bar{\partial}} \underline{\psi}_{-2} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \underline{\psi}_{-k} \cdots. \end{aligned}$$

A map f from a Riemannian manifold M to FP^n is linearly full if the image of f does not lie in any hyperplanes of FP^n , F could be $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

We say that the harmonic ψ strongly isotropic if for every positive integer k holds $\psi \perp \psi_k$. Otherwise, there exists a maximal integer r satisfying for every $0 < i \leq r, \psi_i \perp \psi$. The integer r is referred to as the isotropic order of ψ .

Definition 2.2 Let ψ be a minimal sphere in $G(k, n; \mathbb{C})$ with isotropic order l . We define the first ∂ return of ψ as

$$c'_l(\psi) = A'_{\psi_l, \psi} \circ A'_{\psi_{l-1}, \psi_l} \circ \cdots \circ A'_{\psi, \psi_l}.$$

Suppose that $\phi: M^2 \rightarrow G(k, n; \mathbb{C})$ is a harmonic map. The map ϕ is said to be ∂ -irreducible if $\text{rank } \phi = \text{rank } \phi_1$, where $\phi_1 = A'_{\phi, \phi^\perp}$; the map ϕ is said to be ∂ -reducible if $\text{rank } \phi > \text{rank } \phi_1$. We can define $\bar{\partial}$ reducible/irreducible similarly.

Let $\psi: M \rightarrow \mathbb{C}P^n$ be a smooth map. Then we can identify the smooth map ψ with a subbundle of rank one of the trivial bundle $\underline{\mathbb{C}^{n+1}} = M \times \mathbb{C}^{n+1}$,

$$\underline{\psi} = \{(p, \gamma) | p \in M, \gamma \in \psi(p)\} \subset M \times \mathbb{C}^{n+1}.$$

Therefore, the fiber of ψ at p is $\underline{\psi}|_p = \psi(p)$. When ψ is a harmonic map, we also call the corresponding subbundle $\underline{\psi}$ harmonic. Let L denote the canonical line subbundle of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$, i. e., $L = \{(p, l) | p \in \mathbb{C}P^n, l \in p\}$. Then $\underline{\psi} = \psi^{-1}L$. Conversely, any sub-bundle of rank 1 of $\mathbb{C}^{n+1}$ can induce a smooth map from M to $\mathbb{C}P^n$.

Let $\underline{\phi}, \underline{\psi}$ be two subbundles of $\mathbb{C}^n$ that are perpendicular to each other. Under the local coordinate z , we define ∂ -second fundamental form

$$A'_{\underline{\phi}, \underline{\psi}^\perp}: \underline{\phi} \rightarrow \underline{\psi}^\perp, \nu \mapsto \pi_{\underline{\psi}^\perp} \left(\frac{\partial \nu}{\partial z} \right),$$

where ν is a local smooth section of $\underline{\psi}$, $\pi_{\underline{\psi}^\perp}$ is the orthogonal projection with respect to $\underline{\psi}$. Similarly, we can define $\bar{\partial}$ -second fundamental form $A''_{\underline{\phi}, \underline{\psi}^\perp}$.

Suppose that $\psi: S^2 \rightarrow \mathbb{C}P^n$ is a linearly full conformal harmonic immersion, ψ can be induced by a linearly full holomorphic map $\psi_0: S^2 \rightarrow \mathbb{C}P^n$, where ψ_0 generates the following harmonic sequence^[2]:

$$\underline{\psi}_0 \xrightarrow{\partial} \underline{\psi}_1 \xrightarrow{\partial} \underline{\psi}_2 \cdots \xrightarrow{\partial} \underline{\psi}_n \xrightarrow{\partial} 0.$$

Here, ψ is linearly full, that is, ψ does not lie in any hyperplane of $\mathbb{C}P^n$. Let z be the local complex coordinate of S^2 (for $z_0, z_1 \in \mathbb{C}, z_0 \neq 0$, and $[(z_0, z_1)^T] \in \mathbb{C}P^1, z = \frac{z_1}{z_0}$ and choose a local holomorphic section f_0 of the vector bundle $\underline{\psi}_0$, i. e., $\frac{\partial f_0}{\partial \bar{z}} = 0$. The local sections f_{p+1} of $\underline{\psi}_{p+1}$ can be defined by the following formula:

$$\frac{\partial f_p}{\partial \bar{z}} = f_{p+1} + \frac{\langle \frac{\partial f}{\partial \bar{z}}, f_p \rangle}{|f_p|^2} f_p, 0 \leq p \leq n-1, \quad (4)$$

where $|f_p|^2 = \langle f_p, f_p \rangle$, with $\langle \cdot, \cdot \rangle$ the Hermitian inner product. Besides, we have

$$\frac{\partial f_p}{\partial \bar{z}} = -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}, 1 \leq p \leq n. \quad (5)$$

Let $V_p^n: S^2 \rightarrow \mathbb{C}P^n$,

$$z \mapsto \left[(g_{p,0}(z), g_{p,1}(z), \cdots, g_{p,n}(z))^T \right].$$

where $p = 0, 1, \cdots, n$, and for $r = 0, 1, \cdots, n$,

$$\begin{aligned} g_{p,r}(z) &= \frac{p!}{(1+z\bar{z})^p} \sqrt{\binom{n}{r}} z^{r-p}, \\ \sum_k (-1)^k \binom{r}{p-k} \binom{n-r}{k} (z\bar{z})^k. \end{aligned}$$

Let

$$g_p = (g_{p,0}(z), \cdots, g_{p,n}(z))^T,$$

that is, $V_p^n = [g_p]$. By calculation, we have following results:

$$g_{p+1} = -\frac{\langle \frac{\partial g}{\partial \bar{z}}, g_p \rangle}{|g_p|^2} g_p + \frac{\partial g_p}{\partial \bar{z}}, 0 \leq p \leq n-1,$$

$$|g_p|^2 = \frac{\partial g}{\partial \bar{z}} = -\frac{|g_p|^2}{|g_{p-1}|^2} g_{p-1},$$

$$\frac{\partial g_p}{\partial \bar{z}} = \frac{p(n-p+1)}{(1+|z|^2)^2} g_{p-1} = -\frac{|g_p|^2}{|g_{p-1}|^2} g_{p-1}.$$

and

$$|g_p|^2 = \frac{n! p!}{(n-p)!} (1+z\bar{z})^{n-2p}. \quad (6)$$

Here

$$g_0(z) = (1, \dots, \sqrt{\binom{n}{n}} z^n)^T. \quad (7)$$

Therefore, V_0^n can generate a harmonic sequence

$$0 \rightarrow \underline{V_0^n} \xrightarrow{\partial} \underline{V_1^n} \xrightarrow{\partial} \dots \xrightarrow{\partial} \underline{V_n^n} \rightarrow 0.$$

which is called the Veronese sequence of $\mathbb{CP}^n$, and the corresponding map V_p^n ($p = 0, 1, \dots, n$) is called the p -th Veronese map. We equip $\mathbb{CP}^n$ with the Fubini-Study metric of the holomorphic sectional curvature 4, and then let ds_p^2 be the pullback metric of V_p^n in S^2 . We have

$$ds_p^2 = \frac{n+2p(n-p)}{(1+|z|^2)^2} dzd\bar{z}. \quad (8)$$

and its Gaussian curvature $K_p = \frac{4}{n+2p(n-p)}$ by calculation. Therefore, for $p = 0, 1, \dots, n$, V_p^n is a conformal minimal immersion with constant curvature in $\mathbb{CP}^n$. The following proposition is proved by Bolten et al. [2]

Proposition 2.2 If $\psi: S^2 \rightarrow \mathbb{CP}^n$ is a linearly full conformal minimal immersion with constant curvature, then the immersion ψ is an element of the Veronese sequence, up to a holomorphic isometry of $\mathbb{CP}^n$, i. e., there is a unitary matrix $U \in U(n+1)$ and an integer $p \in \{0, 1, \dots, n\}$ such that

$$\psi = U \cdot V_p^n = [U \cdot g_p^n]. \quad (9)$$

Hence, by Lemma 1.3 and Proposition 2.1, we have:

Proposition 2.3 Let $\psi: S^2 \rightarrow \mathbb{CP}^{2n+1}$ be an element of the Veronese sequence in $\mathbb{CP}^{2n+1}$. If there exists a $U \in U(2n+2)$ such that $U \cdot \psi$ is horizontal in $\mathbb{CP}^{2n+1}$, then $\pi \circ (U \cdot \psi): S^2 \rightarrow \mathbb{HP}^n$ is a conformal minimal two-sphere with constant curvature, where $\pi: \mathbb{CP}^{2n+1} \rightarrow \mathbb{HP}^n$ is the twistor map.

Here and in the sequel: we denote the matrix $F \in M_{m+1}$ with entries

$$F_{i,j} = 0, i+j \neq m+2,$$

by

$$F = \text{antidiag}\{F_{1,m+1}, \dots, F_{m+1,1}\}. \quad (10)$$

We have the following lemma (It was proved by Jiao and Cui [8] when n is even, here we prove in fact that this statement holds when $n \in \mathbb{N}^+$):

Lemma 2.1

$$V_p^n = J_{n+1} \overline{V_{n-p}^n}, n \in \mathbb{N}^+, 0 \leq p \leq n, \quad (11)$$

where

$$J_{n+1} = \text{antidiag}\{(-1)^n, \dots, (-1)^0\}. \quad (12)$$

Proof Recall that $V_p^n = [g_p^n]$, where g_p^n is the homogeneous coordinate V_p^n . We are going to prove $[g_p^n] = J_{n+1} [\overline{g_{n-p}^n}]$. We have already defined $[g_p^n]$, therefore,

$$g_{n-p,n-r}(z) = \frac{(n-p)!}{(1+z\bar{z})^{n-p}} \sqrt{\binom{n}{n-p}} z^{p-r} \sum_l (-1)^l \binom{n-r}{n-p-l} \binom{r}{l} (z\bar{z})^l.$$

Let $l=r-p+k$ and collect terms, we have

$$g_{p,r}(z) = (-1)^{r-p} \frac{(n-p)!}{p!(1+z\bar{z})^{n-2p}} \overline{g_{n-p,n-r}},$$

and the proof is complete. $\square$

3 Construction of horizontal minimal 2-sphere

Theorem 3.1 For $\Psi = [f] = [(f_1, f_2, \dots, f_{2n+2})]: S^2 \rightarrow \mathbb{CP}^{2n+1}$, if $U \in U(2n+2)$ Then $U\Psi$ is horizontal if and only if

$$\sum_{i,j=1}^{2n+2} A_{ij} f_i \partial f_j = 0, \quad (13)$$

$$\sum_{i,j=1}^{2n+2} A_{ij} f_i \bar{\partial} f_j = 0. \quad (14)$$

which can be rewritten in a matrix form

$$f^T A \partial f = 0,$$

$$f^T A \bar{\partial} f = 0.$$

$$\partial f = (\partial f_1, \dots, \partial f_{2n+2})^T, \bar{\partial} f = (\bar{\partial} f_1, \dots, \bar{\partial} f_{2n+2})^T,$$

where we fix the notation that $A = (A_{ij}) = U^T \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix} U = U^T B_{n+1} U$, z is the local holomorphic coordinate of S^2 .

This form of horizontal condition first appeared in Chen and Jiao [6], however, it can be proved directly from the definition of horizontal map and the fact that π is a Riemannian submersion.

Theorem 3.2 If equation $U^T B_{n+1} U = A$ has two solutions U_1 and U_2 , there exists $M \in Sp(n)$

such that $U_1 M = U_2$.

Conversely, if $U_1^T B_{n+1} U_1 = A$, then for arbitrary $M \in Sp(n)$, $(MU_1)^T B_{n+1} MU_1 = A$.

According to the Theorem 3.2, it is sufficient to find a special solution of the equation $U^T B_{n+1} U = A$, since the rest of the solutions are symplectic equivalent to the special solution under the twistor map.

We assume that A has the form of $\begin{pmatrix} 0 & A_1^t \\ -A_1 & 0 \end{pmatrix}$.

Suppose that U has the form as follows:

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & I_{n+1} \end{pmatrix}.$$

Then $U_1 = A_1$.

Fei and He^[16] found the trivial solution $U_{\text{triv}}^T B_{n+1} U_{\text{triv}} = J_{2n+2}$ that makes $U_{\text{triv}} V_p^{2n+1}$, $p = 1, \dots, n-1$ horizontal.

Our first job is to find other U such that UV_0^{2n+1} is horizontal.

We observe (14) holds trivially, therefore, it is sufficient to solve (13). Each equation of the equation system (13) is a polynomial equation of variable z . The coefficients of the term of polynomial should vanish, so we have the following equations:

$$\begin{aligned} & A_{1,2} = 0, A_{1,3} = 0, \dots, \\ & k \sqrt{\binom{2n+1}{0}} \sqrt{\binom{2n+1}{k}} A_{1,k+1} + \\ & (k-1) \sqrt{\binom{2n+1}{1}} \sqrt{\binom{2n+1}{k-1}} A_{2,k} + \dots + \\ & \sqrt{\binom{2n+1}{k-1}} \sqrt{\binom{2n+1}{1}} A_{k,2} = 0, \dots, \quad (15) \\ & (2n+1) \cdot \binom{2n+1}{0} A_{1,2n+2} + 2n \binom{2n+1}{1} A_{2,2n+1} + \\ & \dots + n+1 \binom{2n+1}{n+1} A_{n+1,n+2} + \dots + \\ & \binom{2n+1}{2n} A_{2n+1,2} = 0, \dots, \\ & ((2n+1) \sqrt{\binom{2n+1}{l}} \sqrt{\binom{2n+1}{2n+1}} A_{l+1,2n+2} + \\ & \dots + \sqrt{\binom{2n+1}{2n+1}} \sqrt{\binom{2n+1}{l}} A_{2n+2,l+1} = 0, \dots, \\ & A_{2n-1,2n+2} = 0, A_{2n,2n+2} = 0. \end{aligned}$$

Notice that A is a skew-symmetric matrix; then we need to solve

$$\sum_{i=1}^{n+1} (2n+3-2i) \binom{2n+1}{i-1} A_{i,2n+3-i} = 0 \quad (16)$$

We assume A has nonzero elements only in the antidiagonal, that is, we set the elements $A_{k,l} = 0, k+l \neq 2n+3$.

Notice that A is a unitary matrix, the length of complex numbers $A_{m,2n+3-m}, 1 \leq m \leq 2n+2$ are equal to 1. Denote the coefficients of $A_{k,2n+3-k}$ as F_k , the vector $F_k A_{k,2n+3-k}, 1 \leq k \leq n+1$ by B_k , respectively. Then the equation (16) can be written as $\sum_{i=1}^{2n+2} B_k = 0$. The solution to the equation (16) exists if and only if $B_1, \dots, B_{n+1}$ can form an $n+1$ -polygon. Now we need the following elementary yet useful theorem which appeared in Zhang and Jiao^[9].

Theorem 3.3 Given $n (n \geq 2)$ segments of any arbitrary length, these segments can form an $n+1$ (convex) polygon if and only if for an arbitrary segment in these segments, its length is less or equal to the sum of length of the n remaining segments.

Remark The segments may be collinear, in this case, the length of one segment can be equal to the sum of length of the remaining segments. The claim holds in these degenerate cases, hence in the sequel, we use this fact without mentioning it.

We can prove the Theorem 3.3 by induction, which involves only trigonometry and continuity argument, thus omitted.

We point out to form polygons, when $n=2$, the lengths of the sides must be equal; when $n=3$, the sides satisfy the conditions in the can form a unique triangle; when $n=4$, the sides satisfy the conditions in the Theorem 3.3 can form a 4-polygon with the length of diagonal varies in a closed interval. This argument can be generalized to cases where $n > 4$. We define F_k as

$$F_k = (2n+3-2k) \binom{2n+1}{k-1}, 1 \leq k \leq n+1.$$

Theorem 3.4 There exists unitary matrices $U \in U(2n+2), n \geq 2$ such that the immersion $\pi \circ (U \cdot V_0^{2n+1})$ is a horizontal conformal minimal two-sphere with constant curvature $\frac{4}{2n+1}$ in $\mathbb{HP}^n$.

Proof We consider the sequence of number

$F_k, k = 2, \dots, n$. By definition,

$$\begin{aligned} \frac{F_k}{F_{k-1}} &= \frac{(2n-2k+3) \binom{2n+1}{k-1}}{(2n-2k+5) \binom{2n+1}{k-2}} \\ &= 1 + \frac{4k^2 - 4k(2n+4) + 4n^2 + 14n + 14}{(2n-2k+5)(k-1)}. \end{aligned}$$

In the rest of the proof, we denote $\left\lfloor \frac{2n+4-\sqrt{2n+2}}{2} \right\rfloor$ by K . By the property of quadratic function: if $1 \leq k \leq K$, then $F_k \geq F_{k-1}$; if $K \leq k \leq n+1$, then $F_k \leq F_{k-1}$. Furthermore,

$$\begin{aligned} \frac{F_{k-1}}{F_k} &= \frac{(2n-2k+5) \binom{2n+1}{k-2}}{(2n-2k+3) \binom{2n+1}{k-1}}; \\ \frac{F_{k+1}}{F_k} &= \frac{(2n-2k+1) \binom{2n+1}{k}}{(2n-2k+3) \binom{2n+1}{k-1}}. \end{aligned}$$

By some elementary scaling method for inequalities, we have $\frac{F_{k-1}}{F_k} + \frac{F_{k+1}}{F_k} > 1$. Therefore,

$$F_{k-1} + F_{k+1} > F_k, 2 \leq k \leq n.$$

Then the condition $F_j < \sum_{i \neq j} F_i$ is satisfied.

By Theorem 3.3, there exists a solution to the horizontal equation (13), which completes the proof. $\square$

The existence of linearly full horizontal 2-sphere in $\mathbb{CP}^{2n}$ was proved by Zhang and Jiao^[9].

In the following part, we can set $A_{i, 2n+3-i} = 1$, for some $i \in \{1, \dots, n+1\}$ without loss of generality due to the fact that V_p^{2n+1} is in the space $\mathbb{CP}^{2n+1}$.

Example 3.1 The solution to the horizontal equation $(V_0^5)^T A \partial V_0^5 = 0$ is unique (in the sense of symplectic equivalence) mod $U(1)$ when A is of antidiagonal form (in fact, we can see later that A is indeed an antidiagonal matrix). The only equation we need to solve is $5A_{1,6} + 15A_{2,5} + 10A_{3,4} = 0$, with the length $5 \leq 10 \leq 15$. Thus, it has only one solution $A_{1,6} = 1, A_{2,5} = -1, A_{3,4} = 1$.

The procedure for constructing 2-spheres with desired properties is divided into the following steps:

Step 1. We rearrange F_k in the increasing order of length. It is clear that there exists a

permutation T such that $F_{T(i)} \leq F_{T(i+1)}$ and $F_i \leq F_{T(i)}$. Correspondingly, we have $\tilde{A}_i = A_{T(i), 2n+3-T(i)}$.

Step 2. Let $\tilde{A}_{n+1} = x_1 + y_1 \sqrt{-1}, \tilde{A}_n = x_2 + y_2 \sqrt{-1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. By induction, $d := \sum_{i=1}^{n-2} (-1)^{n-i-1} \tilde{F}_i < \tilde{F}_{n-2}$. Let $\tilde{A}_{i, 2n+3-i} = (-1)^{n-i}, 1 \leq i \leq n-2$, $\tilde{A}_{n-1, n+1} = 1$. By the unitary condition we have $x_1^2 + y_1^2 = 1, x_2^2 + y_2^2 = 1$. By (16) $\tilde{F}_{n+1}x_1 + \tilde{F}_n x_2 + d = 0, \tilde{F}_{n+1}y_1 + \tilde{F}_n y_2 = 0$.

Step 3. Solve the equations in Step 2, we get

$$y_1 = -\frac{\tilde{F}_n}{\tilde{F}_{n+1}} y_2, \quad y_2 = \sqrt{1 - \frac{(F_{n+1}^2 - F_n^2 - d^2)^2}{4F_n^2 d^2}},$$

the existence of y_2 is guaranteed by Theorem 3.3.

If $\tilde{F}_{n+1} = \sum_{i=1}^{n-1} (-1)^{i+1} \tilde{F}_i + \tilde{F}_n$. In this case, we change the sign of $\tilde{A}_2$, that is, we set $\tilde{A}_i = (-1)^{n-1-i}, i \neq 2; \tilde{A}_2 = (-1)^{n-2}$. In this case $x_1, x_2 \neq \pm 1$, which can be checked by some calculation. Therefore, we can always construct holomorphic horizontal 2-spheres with 2 and only 2 elements possessing nonzero imaginary parts in U .

Suppose that $\tilde{U}$ has the form as follows:

$$\tilde{U} = \begin{pmatrix} U_1 & 0 \\ 0 & I_{n+1} \end{pmatrix},$$

where $U_1 \in U(n+1)$.

We assume U satisfies the following equation $\tilde{U}^T B_{n+1} \tilde{U} = \tilde{A}$, where $U_1 = (\text{antidiag} \{(1, -1, 1, -1, \dots, x_2 + y_2 \sqrt{-1}, x_1 + y_1 \sqrt{-1})\})^T$, x_1, x_2, y_1, y_2 are solved by previous procedure. Then

$$\tilde{A} = P_s^T \cdots P_2^T P_1^T A P_1 P_2 \cdots P_s,$$

where P_i are the elementary matrices for exchanging two columns of a matrix. Let $P = P_1 P_2 \cdots P_s$, then

$$\tilde{A} = P^T A P.$$

It is obvious that $P \in \text{Sp}(n+1)$ and the solution to the equation

$$U^T B_{n+1} U = A$$

is $U = P \tilde{U}$.

Theorem 3.5 Following our procedure, we construct horizontal minimal 2-spheres UV_0^{2n+1} , $U \in U(2n+2)$ such that the immersion $\pi \circ (U \cdot V_0^{2n+1})$ is a linearly full conformal minimal two-sphere with constant curvature $\frac{4}{2n+1}$ in $\mathbb{HP}^n$.

Furthermore, there are only two elements on the antidiagonal of U_1 with nonimaginary part.

We have the following examples:

Example 3.2 For $V_0^7 = [(1, \sqrt{7}z, \sqrt{21}z^2, \sqrt{35}z^3, \sqrt{35}z^4, \sqrt{21}z^5, \sqrt{7}z^6, z^7)']$. We need to find special solutions to the equation $7A_{1,8} + 35A_{2,7} + 63A_{3,6} + 35A_{4,5} = 0$ other than the trivial one. Set $A_{1,8} = 1, A_{2,7} = 1$, then solve the equation system: $63A_{3,6} + 35A_{4,5} = -42, |A_{3,6}| = |A_{4,5}| = 1$, we have $A_{3,6} = -\frac{23}{27} - \frac{10\sqrt{2}}{27}\sqrt{-1}, A_{4,5} = \frac{1}{3} + \frac{2\sqrt{2}}{3}\sqrt{-1}$.

The sphere

$$\pi \circ UV_0^7 = \left[\begin{pmatrix} \left(\frac{1}{3} + \frac{2\sqrt{2}}{3}\sqrt{-1}\right)\sqrt{35}z^3 \\ \left(-\frac{23}{27} - \frac{10\sqrt{2}}{27}\sqrt{-1}\right)\sqrt{21}z^2 \\ \sqrt{7}z \\ 1 \end{pmatrix} + j \begin{pmatrix} \sqrt{35}z^4 \\ \sqrt{21}z^5 \\ \sqrt{7}z^6 \\ z^7 \end{pmatrix} \right]_{\mathbb{H}}$$

with constant curvature $2/7$ is

horizontal in $\mathbb{HP}^3$. For convenience, we denote $A_{2,7} = x + w\sqrt{-1}, A_{3,6} = y + z\sqrt{-1}, A_{4,5} = f + g\sqrt{-1}$. Actually, by our analysis, the solution is parametrized by y (or x, w, f, g, z , the result is essentially the same) when y is in some interval. We give the expression of f in terms of y , other terms can be solved subsequently.

$$f = \frac{-81y^2 + 27\sqrt{c} + 378y - 41}{10(9y - 41)}, -1 < y \leq \frac{23}{27},$$

$$c = 9y^4 - 32y^3 - 50y^2 + 32y + 41.$$

Example 3.3 For $V_0^9 = [(1, \sqrt{9}z, \sqrt{36}z^2, \dots, \sqrt{36}z^7, \sqrt{9}z^8, z^9)^T]$, we have found a special solution: $A_{1,10} = 1, A_{2,9} = 1, A_{3,8} = -\frac{5}{44} + \frac{7\sqrt{39}}{44}\sqrt{-1}, A_{4,7} = -\frac{31}{44} - \frac{5\sqrt{39}}{44}\sqrt{-1}, A_{5,6} = 1$. Thus we have a nontrivial solution in $\mathbb{CP}^9$, we construct a new conformal minimal 2-sphere with constant curvature $4/9$ in $\mathbb{HP}^4$.

Example 3.4 For V_0^{11} we have found another

non-trivial solution $A = \begin{pmatrix} 0 & A_1 \\ -A_1^T & 0 \end{pmatrix}$, where $A_1 = \text{antidiag} \{-1, 1, \frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{-1}, -\frac{\sqrt{3}}{2} + \frac{\sqrt{1}}{2}\sqrt{-1}, x_1 + y_1\sqrt{-1}, x_2 + y_2\sqrt{-1}, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sqrt{-1}\}$, x_1, y_1, x_2, y_2 have radical solution. We give the exact radical expression form of y_2 (x_1, y_1, x_2 can be solved in terms of y_2)

$$y_2 = \frac{6\,757\,786 - 3\,068\,604\sqrt{6} - 3\sqrt{3v}}{200(162\,562\sqrt{2} - 123\,831\sqrt{3})},$$

$$v = 53\,118\,796\,990\,549 - 21\,626\,305\,201\,536\sqrt{6}.$$

Here, we give the approximate value: $y_2 \approx -0.881\,413$; $x_1 \approx 0.593\,697$; $y_1 \approx -0.472\,341$; $x_2 \approx 0.804\,689$. Thus, we construct a new conformal minimal 2-sphere with constant curvature $4/11$ in $\mathbb{HP}^5$.

Example 3.5 For V_0^{13} we find a special solution to the following equation $13A_{1,14} + 143A_{2,13} + 702A_{3,12} + 2\,002A_{4,11} + 3\,575A_{5,10} + 3\,861A_{6,9} + 1\,716A_{7,8} = 0$. First, we compare the coefficient: $13 < 143 < 702 < 1\,716 < 2\,002 < 3\,575 < 3\,861$, then let $A_{1,14} = -1, A_{2,13} = 1, A_{3,12} = -1, A_{7,8} = 1, A_{4,11} = 1$, thus $A_{5,10} = -\frac{19}{55} + \frac{\sqrt{74}}{55}\sqrt{-1}, A_{6,9} = -\frac{49}{99} - \frac{10\sqrt{74}}{99}\sqrt{-1}$. Then we can solve the corresponding U , such that $U^T B_7 U = A$, and then $\pi \circ UV_0^{13}$ is a conformal minimal 2-sphere in $\mathbb{HP}^6$ with constant curvature $4/13$.

To illustrate our principle applying in general case, we give the following example.

Example 3.6 In the case of V_0^{25} , we give a special solution to the horizontal equation: $A_{i,27-i} = (-1)^i, i \leq 7, A_{13,14} = 1, A_{8,19} = -1, A_{9,18} = 1, A_{12,15} = 1, A_{11,16} = -\frac{180\,781}{243\,474} - \frac{\sqrt{26\,597\,818\,715}}{243\,474}\sqrt{-1}, A_{10,17} = -\frac{548\,279}{852\,159} + 4\frac{\sqrt{26\,597\,818\,715}}{852\,159}$. Then we can solve the corresponding U such that $U^T B_{13} U = A$, therefore $\pi \circ UV_0^{25}$ is a conformal minimal 2-sphere in $\mathbb{HP}^{12}$ with constant curvature $4/25$.

The horizontal spheres in $\mathbb{CP}^{2n-1}$ can be embedded in $\mathbb{HP}^{n-1}$. Therefore, they are not linearly full in $\mathbb{HP}^n$. In the following part, we study the conformal minimal horizontal 2-sphere with

constant curvature and linearly full in $\mathbb{CP}^{2n}$.

We construct minimal 2-spheres in $\mathbb{HP}^n$ by the following process:

$$f: S^2 \rightarrow \mathbb{CP}^{2n} \xrightarrow{i} \mathbb{CP}^{2n+1} \xrightarrow{\pi} \mathbb{HP}^n.$$

Suppose that the 2-sphere f is a conformal minimal sphere with constant curvature linearly full in $\mathbb{CP}^{2n}$. By the Proposition 2.2, there exists a unitary matrix $U \in U(2n+1)$ such that $f = UV_p^{2n}$, then we embed $f \in \mathbb{CP}^{2n}$ to $\tilde{f} \in \mathbb{CP}^{2n+1}$ by appending zero to the end of the homogeneous coordinate of f .

Two solutions to the same horizontal equation are symplectic equivalent. We assume

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & I_{n+2} \end{pmatrix},$$

$$U_1 = \text{antidiag}\{U_{1,n}, U_{2,n-1}, \dots, U_{n,1}\}.$$

The horizontal equation of f is

$$\tilde{f}^i B_{n+1} \partial \tilde{f} = 0.$$

To construct a horizontal sphere UV_0^{2n} , it is sufficient to give a special to the following equation:

$$\sum_{i=1}^{n-1} (2n+2-2i) \binom{2n}{i} U_{2n+1-i,i} = 0. \quad (17)$$

Theorem 3.6 We construct minimal 2-sphere linearly full in $\mathbb{HP}^n$ with constant curvature $2/n$ explicitly. Furthermore, there are two and only two elements of U_1 have non-zero imaginary part.

The procedure of constructing horizontal minimal 2-sphere is similar to Theorem 3.5, we leave the proof to the reader.

We have the following examples to illustrate our method.

Example 3.7 Suppose that $U \cdot V_0^{10}$ is a horizontal conformal minimal 2-sphere linearly full in $\mathbb{CP}^{2n}$ with the special form. It is sufficient to solve

$$U_{5,1} + 8U_{4,2} + 27U_{3,3} + 48U_{2,4} + 42U_{1,5} = 0.$$

The largest coefficient is before $U_{2,4}$, so we set $U_{5,1} = 1, U_{4,2} = -1, U_{1,5} = 1$, then $U_{3,3} = \frac{47}{96} + \frac{7\sqrt{143}}{96}\sqrt{-1}$, $U_{2,4} = -\frac{1}{12} - \frac{\sqrt{143}}{12}\sqrt{-1}$. As a result, we have constructed a holomorphic horizontal minimal 2-sphere in $\mathbb{HP}^5$ with constant curvature $2/5$.

Example 3.8 Similarly, suppose that UV_0^8 is horizontal, where U has the special form. It is

sufficient to solve

$$U_{4,1} + 6U_{3,2} + 14U_{2,3} + 14U_{1,4} = 0.$$

We give a special solution to the equation;

$$U_{4,1} = \frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}; \quad U_{3,2} = \frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{-1};$$

$$U_{2,3} = x_1 + y_1\sqrt{-1}; \quad U_{1,4} = x_2 + y_2\sqrt{-1}.$$

As mentioned before, once we solve y_2 , the variables x_1, x_2, y_1 can be solved by straight forward computation. Here,

$$y_2 = \frac{1554 + 1513\sqrt{3} - \sqrt{3}w}{56(1 + 6\sqrt{3})}.$$

$$w = (124140561 + 41491492\sqrt{3}).$$

Therefore, we have constructed a conformal minimal sphere $\pi \circ UV_0^8$ with constant curvature $\frac{1}{2}$.

When $U_{4,1} = 1$, the solution of the equation (16) mod $U(1)$ is actually parametrized by the imaginary part of $U_{3,2}$ denote by y_2 , then $y_2 = \frac{-56y^2 + 14\sqrt{16y^4 + 40y^3 - 567y^2 - 40y + 551} + 718y - 51}{4(28y - 197)}$,

$-1 < y \leq -47/52$, which means for every y in the interval $-1 < y \leq -47/52$, there exists a unique solution to the equation where x_1, y_1, x_2 can be solved subsequently.

Example 3.9 Suppose UV_0^{12} is horizontal, where U has the special form. It is sufficient to solve $U_{6,1} + 10U_{5,2} + 44U_{4,3} + 110U_{3,4} + 165U_{2,5} + 132U_{1,6} = 0$.

Since $1 < 10 < 44 < 110 < 132 < 165$, we set $U_{6,1} = 1, U_{5,2} = -1, U_{4,3} = 1, U_{3,4} = 1$, then $U_{2,5} = -\frac{1403}{4785} + \frac{2\sqrt{5231954}}{4785}\sqrt{-1}$, $U_{1,6} = -\frac{15413}{23925} - \frac{8\sqrt{5231954}}{23925}\sqrt{-1}$. As a result, we have

constructed a horizontal holomorphic minimal sphere in $\mathbb{HP}^6$ with constant curvature $1/3$.

Example 3.10 Suppose UV_0^{26} is a horizontal 2-sphere, it is sufficient to solve to find a solution to the following equation:

$$\begin{aligned} & 26U_{13,1} + 624U_{12,2} + 7150U_{11,3} + 52000U_{10,4} + \\ & 269100U_{9,5} + 1052480U_{8,6} + 3223220U_{7,7} + \\ & 7893600U_{6,8} + 15622750U_{5,9} + 24996400U_{4,10} + \\ & 31870410U_{3,11} + 30904640U_{2,12} + \\ & 19315400U_{1,13} = 0. \end{aligned}$$

Compare the coefficients: $26 < 624 < 7\,150$
 $< 52\,000 < 269\,100 < 10\,524\,803\,223\,220 < 7\,893\,600$
 $< 15\,622\,750 < 19\,315\,400 < 24\,996\,400 < 30\,904\,640$
 $< 31\,870\,410$.

We have a special solution: $U_{14-i,i} = (-1)^{10-i}$.
 $1 \leq i \leq 9, U_{1,13} = 1, U_{4,10} = 1, U_{3,11} = -\frac{11\,711\,653}{13\,438\,755} -$
 $\frac{4\sqrt{2\,714\,832\,497\,351}}{13\,438\,755}\sqrt{-1}, U_{2,12} = -\frac{2\,810\,507}{3\,257\,880} +$
 $\frac{\sqrt{2\,714\,832\,497\,351}}{3\,257\,880}$, therefore we construct a
holomorphic 2-sphere with constant curvature $2/13$
and linearly full in $\mathbb{H}\mathbb{P}^{13}$.

Example 3.11 If there exists $U \in U(5)$ such
that $U \cdot V_0^4$ is horizontal, from the equation

$$4U_{2,1} + 8U_{1,2} = 0.$$

The coefficient $4 < 8$, U cannot be in the
special form. Actually, we will prove that UV_0^4
cannot be horizontal by any unitary matrix.

4 Further results by properties of harmonic sequence

The following theorem is an extension of the
main theorem in Section 'The Middle Terms of
Veronese Sequence' of Jiao and Cui^[8]. They
showed that the first four statements are equivalent.
We summarize further equivalent statements into the
following theorem.

Theorem 4.1 Let $\Psi: S^2 \rightarrow \mathbb{C}P^{2n+1}$ be a
linearly full holomorphic map; then Ψ induces the
following harmonic sequence. ($\Psi_i = [f_i]$)

$\Psi = \Psi_0 \rightarrow \Psi_1 \rightarrow \cdots \rightarrow \Psi_{2n+1} \rightarrow 0$ and the
following statements are equivalent ($i \leq n$):

- (1) f_i is horizontal;
- (2) $f_0^t B_{n+1} f_j = 0, 0 \leq j \leq 2i+2$;
- (3) $f_k^t B_{n+1} f_j = 0, 0 \leq k+j \leq 2i+2$;
- (4) f_j is horizontal, $0 \leq j \leq i$;
- (5) $f_0^t B_{n+1} f_j = 0, j = 2i+1$;
- (6) $f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq 2i+1$;
- (7) $f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq 2i+1, j \equiv 1 \pmod{2}$;
- (8) $\partial^s(f_0^t) B_{n+1} \partial^j f_0 = 0, 1 \leq s+j \leq 2i+1$;

When $f_0 = UV_0^{2n+1}$ for some $U \in U_{2n+2}$, denote
 $A = U^T B_{n+1} U$;

$$(9) \sum_{m,k=1}^{2n+2} A_{m,k} k^j \sqrt{\binom{2n+1}{m} \binom{2n+1}{k}} z^{m+k-3-j} = 0,$$

$$1 \leq j \leq 2i+1;$$

$$(10) \sum_{m,k=1}^{2n+2} A_{m,k} k^j \sqrt{\binom{2n+1}{m} \binom{2n+1}{k}} z^{m+k-3-j} = 0,$$

$1 \leq j \leq 2i+1, j \equiv 1 \pmod{2}$. Here we use the
convention that when the power of z is negative, we
let the term with this power equals to zero.

Proof The equivalence of the first four
statements has been proved by Jiao and Cui^[8]. It is
obviously that (2) implies (5). Take $\bar{\partial}$ of (5)
successively and using the property of harmonic
sequence, (5) implies (2).

(2) and (6) are equivalent by taking derivative
and the definition of f_i . (It was proven in
Proposition 5.1 of Jiao and Cui^[8] that $\{V_s^{2n+1}\}$ is the
orthogonal basis constructed from $\{\partial^s V_0^{2n+1}\}$ by
Gram-Schmidt process.)

(6) implies (7) trivially, the inverse direction
can be proved in the following lemma. In the proof
of lemma, we actually prove (6) and (8) are
equivalent.

(6) and (8) are equivalent by collecting
terms. (7) leads to (6) thus implies (9) leads to
(10). The other direction can be proved by using
induction and comparing terms with (7). Thus, we
have proved the theorem. $\square$

Lemma 4.1

$$f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq 2i+1,$$

$$j \equiv 1 \pmod{2} \rightarrow f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq 2i+1.$$

Proof The statement hold when $i=0$.

We actually need to prove $f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j$
 $\leq 2i+1 \rightarrow f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq 2i+2$. Take
the partial derivative of each equation, the only
thing we need to prove is the equation
 $(\partial f_0^t)' B_{n+1} \partial^{2j+1} f_0 = 0$. This could be proved by
using $(\partial^j f_0)' B_{n+1} \partial^{j+1} f_0 = 0$, and taking partial
derivative successively. In the process of induction,
we need to prove $f_0^t B_{n+1} \partial^j f_0 = 0, 1 \leq j \leq l$ is
equivalent to $\partial^s(f_0^t) B_{n+1} \partial^j f_0 = 0, 1 \leq s+j \leq l$, which
can be checked at once.

Theorem 4.1 can be generalized in the
following sense: suppose that $f: S^2 \rightarrow \mathbb{C}P^{2n}$ is a

linearly full holomorphic map, then we denote the natural embedding of f into $\mathbb{CP}^{2n+1}$ by $\tilde{f}$, 0 remain true for $\tilde{f}$.

In the following part, we use Theorem 4.1 together with the fact that $A = U^T B_{n+1} U$ is a skew-symmetric and unitary matrix to get further results.

Define A_k^n as $A_k^n = \frac{n!}{(n-k)!}$, $k \leq n$; $A_k^n = 0$,

$k > n$. We say the elements $A_{i,j}$ such that $i+j=l$, $2 \leq l \leq 2n+3$ are in the $(2n+3-l)$ -th antisuperdiagonal $((2n+3-l)$ -th apd); $i+j=l$, $2n+4 \leq l \leq 4n+4$ are in the $(l-2n-3)$ -th antisubdiagonal $((l-2n-3)$ -th abd), respectively.

Consider the $2n+3-l$ th apd, that is, the entries of $i+j=l$. Assume that UV_p^{2n+1} and l is odd, the apd of $A = U^T B U$ must satisfy a linear equation system $Cx = 0$, here 0 is the zero vector with $l-1$ components, $C_{ij} =$

$$\begin{cases} A_i^{l-j-1}, & i \leq p+1, \\ 1, & i > p+1, j \leq \frac{l-1}{2}, \\ -\frac{\sqrt{\binom{2n+1}{l-i}}}{\sqrt{\binom{2n+1}{i}}}, & i > p+1, j = l-i+p+1, \\ 0, & \text{elsewhere.} \end{cases}$$

the i -th component of the vector x is

$$x_i = \sqrt{\binom{2n+1}{i}} A_{i, l-i}. \text{ Notice that the upper block}$$

of the matrix C involves the permutation number is a part of Vandermonde's matrix, and the rows of this block are linearly independent. If $\frac{l-1}{2} \geq p$, the rank of D_l equals $\frac{l-1}{2} + p + 1$; If $\frac{l-1}{2} < p$ the rank of D_l equals $l-1$.

When l is an even number, observe that the only difference in this case is $A_{\frac{l}{2}, \frac{l}{2}} = 0$. If $\frac{l}{2} \geq p$, the rank of D_l equals $\frac{l}{2} + p$; If $\frac{l}{2} < p$, the rank of D_l equals $l-1$. The analysis for the abd is similar.

When $p=n$, we see that all auds and apds are zero vectors. When $p=n-1$, the matrix is an antidiagonal matrix with one free variable on the

antidiagonal by the unitary condition. Denote the rows of a matrix W by r_i , and the columns by s_j , and there may be some misuse of the notation.

If we have free variables only on the 4th apd, $\dots$ 0th apd and 1th aud $\dots$ 4th aud, then A_{2n+2} is antidiagonal. This statement can be proved by using $r_i \cdot r_j = \delta_{i,j}$ successively. Therefore, we get the following theorem:

Theorem 4.2 When $p = n-3$, the corresponding A of the horizontal equation is an antidiagonal matrix with three free variables, which is unique mod $U(1)$ in the sense of symplectic equivalence.

We can improve this result by using rank (with respect to the horizontal equation) and unitary condition argument.

The following results are needed^[8]. We give an independent proof:

Theorem 4.3 Suppose that $U^T B_{n+1} U = J_{2n+2}$, then UV_i^{2n+1} , $0 \leq i \leq n-1$ are horizontal.

Proof Suppose that x satisfies $(1+x)^{2n+1} = 0$, taking derivative for $2n-1$ th times successively and let $x = -1$ in these equations, we get a series of combinatorial identities. Equation systems $V_0^{2n+1} J_{2n+2} \partial^s V_0^{2n+1} = 0$, $s \leq 2n-1$ can be verified by the above identities. The result follows from (4) $\leftrightarrow$ (6) in 0.

We have the following theorem:

Theorem 4.4 If $U \in U(2n+2)$ such that UV_{n-3}^{2n+1} is horizontal, then U must satisfy $U^T B U = A = J_{2n+1}$.

Some illustrative examples are given here:

Example 4.1 Suppose that there exists a U such that UV_1^7 is horizontal; then $A = U^T B U$ must satisfy

$$A_{1,8} + 5A_{2,7} + 9A_{3,6} + 5A_{4,5} = 0,$$

and

$$343A_{1,8} + 1505A_{2,7} + 2457A_{3,6} + 1295A_{4,5} = 0.$$

Eliminating $A_{1,8}$ and solving this equation system, we have only one solution mod $U(1)$, that is, the trivial solution $A_{1,8} = 1, A_{2,7} = -1, A_{3,6} = 1, A_{4,5} = -1$.

Example 4.2 We consider the matrix $A = U^T B_{n+1} U$ of V_1^9 such that UV_1^9 is horizontal, then A

must satisfy

$$A_{1,10} + 7A_{2,9} + 20A_{3,8} + 28A_{4,7} + 14A_{5,6} = 0,$$

and

$$729A_{1,10} + 4599A_{2,9} + 12060A_{3,8} + 15876A_{4,7} + 7686A_{5,6} = 0.$$

Eliminate $A_{1,10}$,

$$A_{2,9} + 5A_{3,8} + 9A_{4,7} + 5A_{5,6} = 0.$$

Set $A_{1,10} = 1$, the solution is unique, which implies, if there exists $U \in U(10)$, such that UV_1^9 is horizontal, then $U = J_{10}$, up to the symplectic equivalence mod $U(1)$.

By a similar argument, it is straightforward to prove

Theorem 4.5 $\forall U \in U(2n+1)$, UV_{n-1}^{2n} cannot satisfy the horizontal equation after natural embedding.

Theorem 4.6 If UV_{n-3}^{2n} is horizontal, then

$$A: = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}^T B_{n+1} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \text{ is unique mod } U(1).$$

Example 4.3 Suppose that UV_1^8 is horizontal, then $A: = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}^T B_{n+1} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$ must satisfy

$$A_{1,9} + 6A_{2,8} + 14A_{3,7} + 14A_{4,6} = 0,$$

and

$$32A_{1,9} + 171A_{2,8} + 364A_{3,7} + 343A_{4,6} = 0.$$

Set $A_{1,9}$ to 1, and we find no solution to these equations, which means that $\forall U \in U(2n+1)$, UV_1^8 cannot be rotated horizontally.

Actually, we have the following theorem.

Theorem 4.7 If UV_0^{2n+1} is horizontal, and $A: = U^T B_{n+1} U$ has only two elements with nonzero imaginary part in the procedure in Theorem 3.5, then UV_1^{2n+1} is not a horizontal 2-sphere.

Proof Suppose that this U satisfies the horizontal equation of UV_1^{2n+1} , then from (9) in 0, it must satisfies

$$\begin{aligned} & (2n-2K+3) \binom{2n+2}{K-1} y_1 + \\ & (2n-2K+5) \binom{2n+2}{K-2} y_2 = 0, \end{aligned}$$

and

$$((2n+3-K)^3 - K^3) \binom{2n+2}{K-1} y_1 +$$

$$((2n+4-K)^3 - (K-1)^3)(2n-2K+5) \binom{2n+2}{K-2} y_2 = 0.$$

However, this leads to a contradiction by calculation.

Definition 4.1 A 2-sphere $\phi_0: S^2 \rightarrow \mathbb{H}P^n$ is a homogeneous 2-sphere, if for every two points $p, q \in S^2$, there exists a $\sigma \in \text{SU}(2)$ and a $u \in \text{Sp}(n+1)$ such that $u\phi_0 = \phi_0 \circ \sigma$ and $\sigma(p) = q$

The following quantity gives us a criterion for determining whether the 2-spheres we created are homogeneous or non-homogeneous^[10]:

Theorem 4.8 Suppose that UV_p^d is horizontal but UV_{p+1}^d is not horizontal, $\phi = UV_p^d \oplus JUV_p^d$, and the square of the norm of the first ∂ -return is

$$|c'_{2p+2}(\phi)(z_2)|^2 = \frac{|g(z)|^2}{(2d)^{2p+3}(1+|z|^2)^{2d-4p-6}}, \quad (18)$$

where

$$g(z): = (V_0^d)^T A \frac{\partial^{2p+3} V_0^d}{\partial z^{2p+3}} \quad (19)$$

and $A = U^T B_{n+1} U$.

(When d is an even number, we define the matrix A as

$$A: = \left(\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \right)^T B_{n+1} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 4.9 The squared norm of the image of a homogeneous two-sphere's first ∂ return with finite isotropic order is constant.

Proof The only thing we need to prove is that $|c'_{2p+2}(\phi)(z_1)|^2 = |c'_{2p+2}(\phi)(z_2)|^2$, $\forall z_1, z_2 \in S^2$. By the definition of homogeneous sphere, there exists a $\sigma \in \text{SU}(2)$, such that $\sigma(z_1) = z_2$, furthermore, there exists $u \in \text{Sp}(n+1)$, $u \circ g(z_1) = g(z_2)$, u preserve the norm, thus $|g(z_1)|^2 = |g(z_2)|^2$. By the expression of first ∂ return, the proof is complete. $\square$

Example 4.4 UV_0^7 is horizontal, where $A = U^T B_4 U$ satisfies $A = \text{antidiag}\{1, 1, -\frac{23}{27} - \frac{10\sqrt{2}}{27}\sqrt{-1}, \frac{1}{3} + \frac{2\sqrt{2}}{3}\sqrt{-1}, -\frac{1}{3} - \frac{2\sqrt{2}}{3}\sqrt{-1}, \frac{23}{27} + \frac{10\sqrt{2}}{27}\sqrt{-1}, -1, -1\}$, by direct calculation, UV_1^7 is not horizontal, then $|c'_2(UV_0^7 \oplus JUV_0^7)|^2 = \frac{(1125)|z|^8}{(1+|z|^2)^8}$, which is not constant. As a result, this

2-sphere is nonhomogeneous.

Theorem 4.10 The horizontal 2-spheres UV_0^{2n+1} constructed in Theorem 3.5 are not homogeneous. Furthermore, these 2-spheres are reducible and have isotropic order 2.

Proof We have shown that in UV_1^{2n+1} is not horizontal. Then we calculate the square of the norm

$$\text{of first } \partial \text{ return } |c'_2(\phi)|^2 = \frac{|g(z)|^2}{(4n+2)^3(1+|z|^2)^{2n-2}}.$$

Plug U into the following equation

$$g(z) = (V_0^{2n+1})^T A \frac{\partial^3 V_0^{2n+1}}{\partial z^3}, A = U^T B_{n+1} U.$$

Then,

$$g(z) = \sum_{k=1}^{2n+2} \frac{k!}{(k-3)!} \binom{2n+1}{k-1} A_{k, 2n+3-k} z^{2n-2};$$

$$= c|z|^{2n-2}.$$

As a result,

$$|c'_2(\phi)|^2 = \frac{|c|^2 |z|^{4n-4}}{(4n+2)^3 |1+|z|^2|^{4n-4}} \text{ is not constant.}$$

Hence, all these 2-spheres produced by our procedure in Theorem 3.5 are non-homogeneous by Theorem 4.8. $\square$

Similarly, we have the following conclusions.

Example 4.5 Consider U such that $U \cdot V_0^{10}$ is a linearly full holomorphic horizontal 2-sphere in $\mathbb{C}P^{2n}$, then we need to solve

$$U_{5,1} + 8U_{4,2} + 27U_{3,3} + 48U_{2,4} + 42U_{1,5} = 0,$$

we have a special solution $U_{5,1} = 1, U_{4,2} = 1,$

$$U_{3,3} = -1, U_{2,4} = \frac{1}{2} + \frac{\sqrt{3}}{2} \sqrt{-1}, U_{1,5} = -\frac{1}{7} - \frac{4\sqrt{3}}{7} \sqrt{-1},$$

$$U_{6,6} = U_{7,7} = U_{8,8} = U_{9,9} = U_{10,10} = U_{11,11} = 1.$$

By calculation, UV_1^{10} is not horizontal. By direct calculation. The square of the norm of the first partial return is $|c'_2(\phi)|^2 = \frac{969084|z|^{14}}{5(1+|z|^2)^{14}}$ which is not a constant. Here $\phi = UV_0^{10} \oplus JUV_0^{10}$. Therefore, this 2-sphere is non-homogeneous.

Theorem 4.11 The horizontal 2-spheres we construct in Theorem 3.6 are not homogeneous.

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