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一类不确定凸多项式优化的 SOS 松弛对偶问题

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摘要: 考虑一类目标函数和约束函数均具有谱面不确定数据的平方和(SOS)凸多项式优化问题. 首先, 借助 SOS 条件建立带有不确定数据的 SOS 凸多项式系统的择一性定理; 其次, 引入该 SOS 多项式优化问题的 SOS 松弛对偶问题, 并刻画它们之间的鲁棒弱对偶性与强对偶性质; 最后, 借助数值算例说明该 SOS 松弛对偶问题可以重构为半定规划问题.

关键词: SOS 凸多项式; 鲁棒对偶性; 择一性定理
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SOS Relaxation Dual Problem for a Class of Uncertain Convex Polynomial Optimization

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Abstract: We considered a class of sum of squares (SOS) convex polynomial optimization problems with spectrahedral uncertainty data in both objective and constraint functions. Firstly, an alternative theorem for SOS-convex polynomial system with uncertain data was established in terms of SOS conditions. Secondly, we introduced a SOS relaxation dual problem for this SOS polynomial optimization problem and characterized the robust weak and strong duality properties between them. Finally, a numerical example was used to demonstrate that the SOS relaxation dual problem could be reformulated as a semidefinite programming problem.

Keywords: SOS-convex polynomial; robust duality; alternative theorem

SOS 凸多项式在机器学习、信号处理、自动控制和最优电力流等领域应用广泛. 但多项式的凸性检验十分困难, 而多项式的 SOS 凸性可借助求解相应的半定规划(SDP)问题进行检验. 进一步, SOS 凸多项式不仅涵盖了凸二次函数和凸可分离多项式等经典的凸多项式[1-2], 而且也可以是非二次和非分离的多项式. 文献[2-9]给出了 SOS 凸多项式的理论及实际应用的相关成果.

近年来, 关于 SOS 凸多项式结构的不确定性优化问题的鲁棒对偶性研究已得到广泛关注. 借助标量化方法和 SOS 条件, Sun 等[10]研究了目标函数和约束函数均具有谱面不确定数据的多目标 SOS 凸多项式问题的精确 SDP 对偶问题; Jeyakumar 等[11]首先给出了 SOS 凸不等式系统的择一性定理, 并借助 SOS 条件研究了 SOS 凸多项式结构的极大极小优化问题的对偶理论; Jeyakumar 等[12]借助凸集

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强分离定理，刻画了SOS凸不等式系统的择一性定理，并研究了目标函数带有不确定数据的SOS凸多项式优化问题与其对应的SDP松弛对偶问题之间的零对偶间隙性质；Jeyakumar等[13]借助经典的法锥条件，给出了鲁棒SOS凸多项式优化问题的最优性条件，并分别在不确定数据属于多面体不确定集和椭球不确定集的情形下，刻画了该优化问题的精确SDP松弛；Chuong[14]借助SOS条件和线性矩阵不等式，建立了SOS凸多项式结构的不确定性优化问题的鲁棒对偶定理，并将该不确定性优化问题的SOS对偶问题重构为SDP问题；Chuong等[15]借助鲁棒松弛型条件，研究了一类SOS凸多项式结构的两阶段自适应优化问题的精确SDP对偶问题.

受文献[10-12]工作的启发，本文研究一类目标函数和约束函数均具有谱面不确定数据的多目标SOS凸多项式优化问题. 先建立一类不确定SOS凸多项式系统的鲁棒择一性定理，再引入该SOS凸多项式优化问题的SOS松弛对偶问题，并刻画它们之间的鲁棒弱对偶与强对偶性质.

1 预备知识

设$\mathbb{R}^n$为n维欧氏空间，并赋予欧氏范数$\|\cdot\|$. $\mathbb{R}^n_+$表示$\mathbb{R}^n$的非负象限，int $\mathbb{R}^n_+$表示$\mathbb{R}^n_+$的拓扑内部. 记$\boldsymbol{I}_n\in\mathbb{R}^{n\times n}$为$n\times n$阶单位矩阵. 设$S^n$为所有$n\times n$阶实对称矩阵构成的集合. 若对任意的$\boldsymbol{x}\in\mathbb{R}^n$，$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}\geqslant 0$，则称$\boldsymbol{M}\in S^n$是半正定矩阵，记为$\boldsymbol{M}\succeq 0$. $\mathbb{R}[\boldsymbol{x}]$表示$\mathbb{R}^n$上所有实多项式组成的空间，$\mathbb{R}[\boldsymbol{x}]_d$表示$\mathbb{R}^n$上所有次数不超过$d$的全体实多项式组成的空间. $\deg f$表示多项式f的次数. 若存在实多项式$f_r(r=1,2,\cdots,p)$，使得$f=\sum\limits_{r=1}^{p}f_r^2$，则称实多项式$f$是SOS多项式. 对任意的$\boldsymbol{x}\in\mathbb{R}^n$，所有次数不超过$d$的SOS多项式组成的集合记为$\Sigma_d^2[\boldsymbol{x}]$.

定义1[3] 设f为$\mathbb{R}^n$上的实多项式，$\nabla f(\boldsymbol{y})$为f在$\boldsymbol{y}\in\mathbb{R}^n$处的梯度. 若多项式

$$f(\boldsymbol{x})-f(\boldsymbol{y})-\nabla f(\boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{y})$$

在$\mathbb{R}[\boldsymbol{x};\boldsymbol{y}]$(关于变量$\boldsymbol{x}$和$\boldsymbol{y}$)上是SOS多项式，则称多项式$f$在$\mathbb{R}^n$上是SOS凸多项式.

注1 显然，SOS凸多项式是凸多项式，反之不成立[2,5]. 此外，凸二次函数和凸可分离多项式也是SOS凸多项式.

命题1[11] 任意非负的SOS凸多项式一定是SOS多项式.

设$\boldsymbol{A}_r^i(i=0,1,\cdots,m;\ r=1,2,\cdots,p)$和$\boldsymbol{B}_l^j(j=0,1,2,\cdots,s;\ l=1,2,\cdots,k)$是给定的$n\times n$阶实对称矩阵，设$\Omega_r(r=1,2,\cdots,p)$和$\Theta_l(l=1,2,\cdots,k)$为谱面体[16-17]，即

$$\Omega_r:=\left\{\boldsymbol{u}_r:=(u_r^1,\cdots,u_r^m)\in\mathbb{R}^m\,\middle|\,\boldsymbol{A}_r^0+\sum_{i=1}^{m}u_r^i\boldsymbol{A}_r^i\succeq 0\right\},\tag{1}$$

$$\Theta_l:=\left\{\boldsymbol{v}_l:=(v_l^1,\cdots,v_l^s)\in\mathbb{R}^s\,\middle|\,\boldsymbol{B}_l^0+\sum_{j=1}^{s}v_l^j\boldsymbol{B}_l^j\succeq 0\right\}.\tag{2}$$

该谱面体包含鲁棒优化问题中常见的多面体、球、椭球和圆柱体等不确定集合[17]. 不失一般性，本文假设$\Omega_r(r=1,2,\cdots,p)$和$\Theta_l(l=1,2,\cdots,k)$是非空有界集合.

本文考虑如下不确定多目标SOS凸多项式优化问题：

$$(\mathrm{UP})\quad\min_{\boldsymbol{x}\in\mathbb{R}^n}\{(f_1(\boldsymbol{x},\boldsymbol{u}_1),\cdots,f_p(\boldsymbol{x},\boldsymbol{u}_p))\,|\,g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ l=1,2,\cdots,k\},$$

其中：$f_r:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$和$g_l:\mathbb{R}^n\times\mathbb{R}^s\to\mathbb{R}$是给定的函数；$\boldsymbol{u}_r(r=1,2,\cdots,p)$和$\boldsymbol{v}_l(l=1,2,\cdots,k)$为不确定参数，分别属于不确定集$\Omega_r\subset\mathbb{R}^m$和$\Theta_l\subset\mathbb{R}^s$.

若无特殊说明，本文假设对每个固定的$\boldsymbol{u}_r\in\Omega_r$和$\boldsymbol{v}_l\in\Theta_l$，$f_r(\cdot,\boldsymbol{u}_r)$和$g_l(\cdot,\boldsymbol{v}_l)$是SOS凸多项式. 进一步，假设对每个固定的$\boldsymbol{x}\in\mathbb{R}^n$，$f_r(\boldsymbol{x},\cdot)$和$g_l(\boldsymbol{x},\cdot)$是仿射函数，即

$$f_r(\boldsymbol{x},\boldsymbol{u}_r):=b_r^0(\boldsymbol{x})+\sum_{i=1}^{m}u_r^ib_r^i(\boldsymbol{x}),\qquad\forall\,\boldsymbol{u}_r\in\Omega_r,\tag{3}$$

$$g_l(\boldsymbol{x},\boldsymbol{v}_l):=q_l^0(\boldsymbol{x})+\sum_{j=1}^{s}v_l^jq_l^j(\boldsymbol{x}),\qquad\forall\,\boldsymbol{v}_l\in\Theta_l,\tag{4}$$

其中：$b_r^i\in\mathbb{R}[\boldsymbol{x}]_d$，$i=0,1,2,\cdots,m$，$r=1,2,\cdots,p$；$q_l^j\in\mathbb{R}[\boldsymbol{x}]_d$，$j=0,1,2,\cdots,s$，$l=1,2,\cdots,k$.

借助鲁棒优化方法[18-21]，问题(UP)的鲁棒对等问题为

$$(\mathrm{RP})\quad \min_{\boldsymbol{x}\in\mathbb{R}^n}\{(\max_{\boldsymbol{u}_1\in\Omega_1} f_1(\boldsymbol{x},\boldsymbol{u}_1),\cdots,\max_{\boldsymbol{u}_p\in\Omega_p} f_p(\boldsymbol{x},\boldsymbol{u}_p))\,|\,g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ \forall\,\boldsymbol{v}_l\in\Theta_l,\ l=1,2,\cdots,k\}.$$

记 C 为鲁棒对等问题(RP)的可行集，即

$$C:=\{\boldsymbol{x}\in\mathbb{R}^n\,|\,g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ \forall\,\boldsymbol{v}_l\in\Theta_l,\ l=1,2,\cdots,k\}.$$

类似文献[22]中定义 2.24，下面引入问题(UP)鲁棒弱有效解的概念.

定义 2　若 $\bar{\boldsymbol{x}}\in C$ 是问题(RP)的弱有效解，即不存在 $\boldsymbol{x}\in C$，使得

$$\max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)<\max_{\boldsymbol{u}_r\in\Omega_r} f_r(\bar{\boldsymbol{x}},\boldsymbol{u}_r),\qquad r=1,2,\cdots,p,$$

则称 $\bar{\boldsymbol{x}}$ 是问题(UP)的鲁棒弱有效解.

2　鲁棒择一性定理刻画

针对一类具有谱面不确定数据的不确定 SOS 凸多项式系统，下面给出其鲁棒择一性定理. 该结果可视为一般 SOS 凸多项式系统择一性定理的进一步推广.

定理 1　设对任意的 $\boldsymbol{u}_r\in\Omega_r$ 和 $\boldsymbol{v}_l\in\Theta_l$，$f_r(\cdot,\boldsymbol{u}_r)(r=1,2,\cdots,p)$ 和 $g_l(\cdot,\boldsymbol{v}_l)(l=1,2,\cdots,k)$ 均为 SOS 凸多项式. 假设

$$C:=\{\boldsymbol{x}\in\mathbb{R}^n\,|\,g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ \forall\,\boldsymbol{v}_l\in\Theta_l,\ l=1,2,\cdots,k\}\neq\varnothing.$$

则下列结论不能同时成立：

(i) $\exists\,\boldsymbol{x}\in\mathbb{R}^n$，使得 $g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ \forall\,\boldsymbol{v}_l\in\Theta_l,\ l=1,2,\cdots,k\Rightarrow\max\limits_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)<0,\ r=1,2,\cdots,p$；

(ii) $\forall\,\varepsilon>0$，$\exists\,\alpha_r^0\in\mathbb{R}_+$，$\sum\limits_{r=1}^{p}\alpha_r^0=1$，$\alpha_r^i\in\mathbb{R}$，$i=1,2,\cdots,m$，$\lambda_l^0\in\mathbb{R}_+$，$\lambda_l^j\,\mathbb{R}$，$j=1,2,\cdots,s$，$\sigma_0\in\Sigma_d^2[\boldsymbol{x}]$，使得

$$\sum_{r=1}^{p}\Big(\alpha_r^0 b_r^0+\sum_{i=1}^{m}\alpha_r^i b_r^i\Big)+\sum_{l=1}^{k}\Big(\lambda_l^0 q_l^0+\sum_{j=1}^{s}\lambda_l^j q_l^j\Big)+\varepsilon=\sigma_0,\tag{5}$$

$$\alpha_r^0\boldsymbol{A}_r^0+\sum_{i=1}^{m}\alpha_r^i\boldsymbol{A}_r^i\succeq 0,\quad \lambda_l^0\boldsymbol{B}_l^0+\sum_{j=1}^{s}\lambda_l^j\boldsymbol{B}_l^j\succeq 0,\quad r=1,2,\cdots,p,\quad l=1,2,\cdots,k.\tag{6}$$

证明：假设结论(ii)成立，则对任意的 $\varepsilon>0$，存在 $\alpha_r^0\in\mathbb{R}_+$，$\sum\limits_{r=1}^{p}\alpha_r^0=1$，$\alpha_r^i\in\mathbb{R}(i=1,2,\cdots,m)$，$\lambda_l^0\in\mathbb{R}_+$，$\lambda_l^j\in\mathbb{R}(j=1,2,\cdots,s)$ 和 $\sigma_0\in\Sigma_d^2[\boldsymbol{x}]$，使得式(5)和式(6)成立. 类似文献[14]中定理 2.3 的证明方法，由式(5)和式(6)可知，存在 $\hat{\boldsymbol{u}}_r\in\Omega_r(r=1,2,\cdots,p)$ 和 $\hat{\boldsymbol{v}}_l\in\Theta_l(l=1,2,\cdots,k)$，使得

$$\sum_{r=1}^{p}\alpha_r^0 f_r(\boldsymbol{x},\hat{\boldsymbol{u}}_r)+\sum_{l=1}^{k}\lambda_l^0 g_l(\boldsymbol{x},\hat{\boldsymbol{v}}_l)+\varepsilon=\sigma_0(\boldsymbol{x}),\qquad\forall\,\boldsymbol{x}\in\mathbb{R}^n.$$

显然，

$$\sum_{r=1}^{p}\alpha_r^0\max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)+\varepsilon\geqslant\sum_{r=1}^{p}\alpha_r^0 f_r(\boldsymbol{x},\hat{\boldsymbol{u}}_r)+\varepsilon=\sigma_0-\sum_{l=1}^{k}\lambda_l^0 g_l(\boldsymbol{x},\hat{\boldsymbol{v}}_l)\geqslant 0,\qquad\forall\,\boldsymbol{x}\in C,$$

其中最后一个不等式成立是因为 $\sigma_0\geqslant 0$ 和 $\boldsymbol{x}\in C$. 又由 $\alpha_r^0\in\mathbb{R}_+$，$\sum\limits_{r=1}^{p}\alpha_r^0=1$ 以及 ε 的任意性可知，存在 $r_0\in\{1,2,\cdots,p\}$，使得

$$\max_{\boldsymbol{u}_{r_0}\in\Omega_{r_0}} f_{r_0}(\boldsymbol{x},\boldsymbol{u}_{r_0})\geqslant 0,\qquad\forall\,\boldsymbol{x}\in C.$$

因此结论(i)不成立.

反之，假设结论(i)不成立，则对任意的 $\varepsilon>0$，下列不等式系统无解：

$$\{\max_{\boldsymbol{v}_l\in\Theta_l} g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ l=1,2,\cdots,k,\ \max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)+\varepsilon\leqslant 0,\ r=1,2,\cdots,p\}.$$

从而由文献[12]中定理 2.6 可知，存在 $\alpha_r^0\in\mathbb{R}_+$，$\sum\limits_{r=1}^{p}\alpha_r^0=1$，$\lambda_l^0\in\mathbb{R}_+$ 和 $\sigma_0\in\Sigma_d^2[\boldsymbol{x}]$，使得

$$\sum_{l=1}^{k}\lambda_l^0\max_{\boldsymbol{v}_l\in\Theta_l} g_l(\boldsymbol{x},\boldsymbol{v}_l)+\sum_{r=1}^{p}\alpha_r^0\max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)+\varepsilon=\sigma_0,\qquad\forall\,\boldsymbol{x}\in\mathbb{R}^n.$$

显然,

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n}\Big\{\sum_{l=1}^{k}\lambda_l^0\max_{\boldsymbol{v}_l\in\Theta_l}g_l(\boldsymbol{x},\boldsymbol{v}_l)+\sum_{r=1}^{p}\alpha_r^0\max_{\boldsymbol{u}_r\in\Omega_r}f_r(\boldsymbol{x},\boldsymbol{u}_r)+\varepsilon\Big\}\geqslant 0.$$

令 $\Omega:=\prod_{r=1}^{p}\Omega_r\subset\mathbb{R}^{pm}$, $\Theta:=\prod_{l=1}^{k}\Theta_l\subset\mathbb{R}^{ks}$, 显然 Ω 和 Θ 是凸紧集. 从而

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n}\max_{\substack{(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_p)\in\Omega,\\(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_k)\in\Theta}}\Big\{\sum_{r=1}^{p}\alpha_r^0f_r(\boldsymbol{x},\boldsymbol{u}_r)+\sum_{l=1}^{k}\lambda_l^0g_l(\boldsymbol{x},\boldsymbol{v}_l)+\varepsilon\Big\}\geqslant 0.$$

故由文献[23]中定理4.2可得

$$\max_{\substack{(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_p)\in\Omega,\\(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_k)\in\Theta}}\inf_{\boldsymbol{x}\in\mathbb{R}^n}\Big\{\sum_{r=1}^{p}\alpha_r^0f_r(\boldsymbol{x},\boldsymbol{u}_r)+\sum_{l=1}^{k}\lambda_l^0g_l(\boldsymbol{x},\boldsymbol{v}_l)+\varepsilon\Big\}\geqslant 0.$$

即存在$(\bar{\boldsymbol{u}}_1,\cdots,\bar{\boldsymbol{u}}_p)\in\Omega$和$(\bar{\boldsymbol{v}}_1,\cdots,\bar{\boldsymbol{v}}_k)\in\Theta$, 使得

$$\sum_{r=1}^{p}\alpha_r^0f_r(\boldsymbol{x},\bar{\boldsymbol{u}}_r)+\sum_{l=1}^{k}\lambda_l^0g_l(\boldsymbol{x},\bar{\boldsymbol{v}}_l)+\varepsilon\geqslant 0,\qquad\forall\,\boldsymbol{x}\in\mathbb{R}^n.\tag{7}$$

又因为 $f_r(\cdot,\boldsymbol{u}_r)(r=1,2,\cdots,p)$和 $g_l(\cdot,\boldsymbol{v}_l)(l=1,2,\cdots,k)$是SOS凸多项式, 所以

$$\sum_{r=1}^{p}\alpha_r^0f_r(\cdot,\bar{\boldsymbol{u}}_r)+\sum_{l=1}^{k}\lambda_l^0g_l(\cdot,\bar{\boldsymbol{v}}_l)+\varepsilon$$

是SOS凸多项式. 因此结合命题1以及式(3),(4),(7)可知, 存在 $\sigma_0\in\Sigma_d^2[\boldsymbol{x}]$, 使得

$$\sum_{r=1}^{p}\Big(\alpha_r^0b_r^0+\sum_{i=1}^{m}\alpha_r^ib_r^i\Big)+\sum_{l=1}^{k}\Big(\lambda_l^0q_l^0+\sum_{j=1}^{s}\lambda_l^jq_l^j\Big)+\varepsilon=\sigma_0,$$

其中: $\alpha_r^i=\alpha_r^0\bar{u}_r^i\in\mathbb{R}$, $r=1,2,\cdots,p$, $i=1,2,\cdots,m$; $\lambda_l^j=\lambda_l^0\bar{v}_l^j\in\mathbb{R}$, $l=1,2,\cdots,k$, $j=1,2,\cdots,s$.

另一方面, 类似文献[14]中定理2.3的证明方法可得式(6)成立. 证毕.

注2 若不确定集合 $\boldsymbol{u}_r\in\Omega_r(r=1,2,\cdots,p)$和 $\boldsymbol{v}_l\in\Theta_l(l=1,2,\cdots k)$为单点集, 则定理1退化为文献[11]中定理2.2和文献[12]中定理2.6.

3 SOS松弛对偶刻画

下面在不借助任何约束规范或正则性条件的前提下, 利用鲁棒择一性定理, 刻画问题(UP)与其SOS松弛对偶问题之间的鲁棒弱对偶和强对偶性质. 首先引入问题(UP)的SOS松弛对偶问题.

设 $\boldsymbol{\omega}:=(\omega_1,\cdots,\omega_p)\in\mathbb{R}^p$, $\boldsymbol{\alpha}:=(\alpha_1^0,\cdots,\alpha_p^0,\alpha_1^1,\cdots,\alpha_p^1,\alpha_1^2,\cdots,\alpha_p^2,\cdots,\alpha_1^m,\cdots,\alpha_p^m)\in\mathbb{R}_+^p\times\mathbb{R}^{pm}$, $\boldsymbol{\lambda}:=(\lambda_1^0,\cdots,\lambda_k^0,\lambda_1^1,\cdots,\lambda_k^1,\lambda_1^2,\cdots,\lambda_k^2,\cdots,\lambda_1^s,\cdots,\lambda_k^s)\in\mathbb{R}_+^k\times\mathbb{R}^{ks}$, 则问题(UP)的SOS松弛对偶问题为

$$(\mathrm{RD})\quad\begin{cases}\max\limits_{\boldsymbol{\omega}\in\mathbb{R}^p,\,\boldsymbol{\alpha}\in\mathbb{R}_+^p\times\mathbb{R}^{pm},\,\boldsymbol{\lambda}\in\mathbb{R}_+^k\times\mathbb{R}^{ks}}\{\omega_1,\cdots,\omega_p\},\\ \text{s.t.}\ \sum\limits_{r=1}^{p}\Big(\alpha_r^0b_r^0+\sum\limits_{i=1}^{m}\alpha_r^ib_r^i\Big)+\sum\limits_{l=1}^{k}\Big(\lambda_l^0q_l^0+\sum\limits_{j=1}^{s}\lambda_l^jq_l^j\Big)-\sum\limits_{r=1}^{p}\alpha_r^0\omega_r\in\Sigma_d^2[\boldsymbol{x}],\\ \alpha_r^0\boldsymbol{A}_r^0+\sum\limits_{i=1}^{m}\alpha_r^i\boldsymbol{A}_r^i\succeq 0,\quad\lambda_l^0\boldsymbol{B}_l^0+\sum\limits_{j=1}^{s}\lambda_l^j\boldsymbol{B}_l^j\succeq 0,\quad r=1,2,\cdots,p,\quad l=1,2,\cdots,k,\quad\sum\limits_{r=1}^{p}\alpha_r^0=1.\end{cases}$$

为刻画它们之间的对偶性性质, 下面引入问题(RD)弱有效解的概念. 为方便, 记问题(RD)的可行集为 $\bar{C}$.

定义3 设$(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{\lambda}})\in\bar{C}$, 若不存在问题(RD)的可行解$(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\bar{C}$, 使得

$$\omega_r>\bar{\omega}_r,\qquad r=1,2,\cdots,p,$$

则称$(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{\lambda}})\in\bar{C}$是问题(RD)的弱有效解.

借助文献[14]中定理3.1的证明方法, 易得问题(UP)和(RD)之间的鲁棒弱对偶性质.

定理2(弱对偶性) 设 $\boldsymbol{x}\in C$, $(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\bar{C}$, 则

$$(\widetilde{F}_1(\boldsymbol{x}),\cdots,\widetilde{F}_p(\boldsymbol{x}))-(\omega_1,\cdots,\omega_p)\notin-\operatorname{int}\mathbb{R}_+^p,$$

其中 $\widetilde{F}_r(\boldsymbol{x}):=\max\limits_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)$，$r=1,2,\cdots,p$.

借助定理 1，可建立问题(UP)和(RD)之间的鲁棒强对偶性质.

定理 3(强对偶性)　设 $C\neq\varnothing$，若 $\overline{\boldsymbol{x}}\in C$ 是问题(UP)的鲁棒弱有效解，则存在 $\overline{\boldsymbol{\omega}}\in\mathbb{R}^p$，$\overline{\boldsymbol{\alpha}}\in\mathbb{R}^p_+\times\mathbb{R}^{pm}$，$\overline{\boldsymbol{\lambda}}\in\mathbb{R}^k_+\times\mathbb{R}^{ks}$，使得$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})$为问题(RD)的弱有效解，且满足$(\overline{\omega}_1,\cdots,\overline{\omega}_p)=(\widetilde{F}_1(\overline{\boldsymbol{x}}),\cdots,\widetilde{F}_p(\overline{\boldsymbol{x}}))$，其中 $\widetilde{F}_r(\overline{\boldsymbol{x}}):=\max\limits_{\boldsymbol{u}_r\in\Omega_r} f_r(\overline{\boldsymbol{x}},\boldsymbol{u}_r)$，$r=1,2,\cdots,p$.

证明：设 $\overline{\boldsymbol{x}}$ 是问题(UP)的鲁棒弱有效解，则不存在 $\boldsymbol{x}\in C$，使得

$$\max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)<\widetilde{F}_r(\overline{\boldsymbol{x}}),\qquad r=1,2,\cdots,p,$$

即

$$\{\max_{\boldsymbol{v}_l\in\Theta_l} g_l(\boldsymbol{x},\boldsymbol{v}_l)\leqslant 0,\ l=1,2,\cdots,k,\ \max_{\boldsymbol{u}_r\in\Omega_r} f_r(\boldsymbol{x},\boldsymbol{u}_r)-\widetilde{F}_r(\overline{\boldsymbol{x}})+\varepsilon\leqslant 0,\ r=1,2,\cdots,p\}=\varnothing.$$

由定理 1 可知，存在 $\overline{\alpha}^0_r\in\mathbb{R}_+$，$\sum\limits_{r=1}^{p}\alpha^0_r=1$，$\overline{\alpha}^i_r\in\mathbb{R}$，$i=1,2,\cdots,m$，$\overline{\lambda}^0_l\in\mathbb{R}_+$，$\overline{\lambda}^j_l\in\mathbb{R}$，$\overline{\sigma}_0\in\Sigma^2_d[\boldsymbol{x}]$，使得

$$\sum_{r=1}^{p}\Big(\overline{\alpha}^0_r b^0_r+\sum_{i=1}^{m}\overline{\alpha}^i_r b^i_r\Big)-\sum_{r=1}^{p}\overline{\alpha}^0_r\widetilde{F}_r(\overline{\boldsymbol{x}})+\sum_{l=1}^{k}\Big(\overline{\lambda}^0_l q^0_l+\sum_{j=1}^{s}\overline{\lambda}^j_l q^j_l\Big)+\varepsilon=\overline{\sigma}_0,$$

$$\overline{\alpha}^0_r\boldsymbol{A}^0_r+\sum_{i=1}^{m}\overline{\alpha}^i_r\boldsymbol{A}^i_r\succeq 0,\quad \overline{\lambda}^0_l\boldsymbol{B}^0_l+\sum_{j=1}^{s}\overline{\lambda}^j_l\boldsymbol{B}^j_l\succeq 0,\quad r=1,2,\cdots,p,\quad l=1,2,\cdots,k.$$

令 $\overline{\omega}_r:=\widetilde{F}_r(\overline{\boldsymbol{x}})$，$r=1,2,\cdots,p$，$\overline{\boldsymbol{\omega}}:=(\overline{\omega}_1,\cdots,\overline{\omega}_p)\in\mathbb{R}^p$，$\overline{\boldsymbol{\alpha}}:=(\overline{\alpha}^0_1,\cdots,\overline{\alpha}^0_p,\overline{\alpha}^1_1,\cdots,\overline{\alpha}^1_p,\overline{\alpha}^2_1,\cdots,\overline{\alpha}^2_p,\cdots,\overline{\alpha}^m_1,\cdots,\overline{\alpha}^m_p)\in\mathbb{R}^p_+\times\mathbb{R}^{pm}$，$\overline{\boldsymbol{\lambda}}:=(\overline{\lambda}^0_1,\cdots,\overline{\lambda}^0_k,\overline{\lambda}^1_1,\cdots,\overline{\lambda}^1_k,\overline{\lambda}^2_1,\cdots,\overline{\lambda}^2_k,\cdots,\overline{\lambda}^s_1,\cdots,\overline{\lambda}^s_k)\in\mathbb{R}^k_+\times\mathbb{R}^{ks}$. 显然，$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})$ 是问题(RD)的可行解. 故只需证明$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})\in\overline{C}$ 是问题(RD)的弱有效解. 事实上，任取$(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\overline{C}$，由定理 2 可知，$\overline{\omega}_r=\widetilde{F}_r(\overline{\boldsymbol{x}})\geqslant\omega_r(r=1,2,\cdots,p)$ 至少有一个成立. 所以不存在$(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\overline{C}$，使得 $\omega_r>\overline{\omega}_r(r=1,2,\cdots,p)$，从而$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})\in\overline{C}$ 是问题(RD)的弱有效解.

下面给出实例验证定理 2 和定理 3 得到的 SOS 松弛对偶性质.

例 1　考虑问题(UP)，设 $p=m=s=n=2$，$k=1$，令

$$\boldsymbol{A}^0_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \boldsymbol{A}^1_1=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \boldsymbol{A}^2_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\boldsymbol{A}^0_2=\begin{pmatrix}2&0&0\\0&4&0\\0&0&1\end{pmatrix},\quad \boldsymbol{A}^1_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \boldsymbol{A}^2_2=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\boldsymbol{B}^0_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \boldsymbol{B}^1_1=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \boldsymbol{B}^2_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

借助式(1)和式(2)可得

$$\Omega_1:=\{\boldsymbol{u}_1:=(u^1_1,u^2_1)\in\mathbb{R}^2\,|\,(u^1_1)^2+(u^2_1)^2\leqslant 1\},$$

$$\Omega_2:=\left\{\boldsymbol{u}_2:=(u^1_2,u^2_2)\in\mathbb{R}^2\,|\,\frac{(u^1_2)^2}{2}+\frac{(u^2_2)^2}{4}\leqslant 1\right\},$$

$$\Theta_1:=\{\boldsymbol{v}_1:=(v^1_1,v^2_1)\in\mathbb{R}^2\,|\,(v^1_1)^2+(v^2_1)^2\leqslant 1\}.$$

进一步，令 $b^0_1(\boldsymbol{x})=x_1^4-2x_2-1$，$b^1_1(\boldsymbol{x})=1$，$b^2_1(\boldsymbol{x})=0$，$b^0_2(\boldsymbol{x})=x_1^2-x_2-2$，$b^1_2(\boldsymbol{x})=0$，$b^2_2(\boldsymbol{x})=1$，$q^0_1(\boldsymbol{x})=3x_2$，$q^1_1(\boldsymbol{x})=x_1$，$q^2_1(\boldsymbol{x})=x_2$，$\widetilde{F}_1(\boldsymbol{x})=x_1^4-2x_2$，$\widetilde{F}_2(\boldsymbol{x})=x_1^2-x_2$. 则

$$f_1(\boldsymbol{x},\boldsymbol{u}_1)=x_1^4-2x_2-1+u^1_1,\qquad f_2(\boldsymbol{x},u_2)=x_1^2-x_2-2+u^2_2,$$

$$g_1(\boldsymbol{x},\boldsymbol{v}_1)=v^1_1x_1+(v^2_1+3)x_2.$$

从而易得问题(UP)的可行集为

$$C=\{0\}\times(-\infty,0].$$

下面考虑问题问题(RD). 设$(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\overline{C}$, 则

$$\sum_{r=1}^{2}\left(\alpha_r^0 b_r^0+\sum_{i=1}^{2}\alpha_r^i b_r^i\right)+\left(\lambda_1^0 q_1^0+\sum_{j=1}^{2}\lambda_1^j q_1^j\right)-\sum_{r=1}^{2}\alpha_r^0\omega_r\in\Sigma_d^2[\boldsymbol{x}],\tag{8}$$

$$(\alpha_1^1)^2+(\alpha_1^2)^2\leqslant(\alpha_1^0)^2,\quad \frac{(\alpha_2^1)^2}{2}+\frac{(\alpha_2^2)^2}{4}\leqslant(\alpha_2^0)^2,\quad (\lambda_1^1)^2+(\lambda_1^2)^2\leqslant(\lambda_1^0)^2.\tag{9}$$

由式(9)可得

$$\alpha_1^1-\alpha_1^0\leqslant|\alpha_1^1|-\alpha_1^0\leqslant 0,\quad \alpha_2^2-2\alpha_2^0\leqslant|\alpha_2^2|-2\alpha_2^0\leqslant 0,\quad \lambda_1^0+\lambda_1^2\geqslant 0.\tag{10}$$

因为$\boldsymbol{x}\in C$, 所以$(x_1,x_2)\in\{0\}\times(-\infty,0]$. 故由式(8)易得

$$\alpha_1^0(x_1^4-2x_2-\omega_1)+\alpha_2^0(x_1^2-x_2-\omega_2)\geqslant\alpha_1^0-\alpha_1^1+2\alpha_2^0-\alpha_2^2-(3\lambda_1^0x_2+\lambda_1^1x_1+\lambda_1^2x_2)\geqslant 0,$$

其中第二个不等式成立是借助式(10)和$(x_1,x_2)\in\{0\}\times(-\infty,0]$. 因此

$$\alpha_1^0\widetilde{F}_1(\boldsymbol{x})+\alpha_2^0\widetilde{F}_2(\boldsymbol{x})\geqslant\alpha_1^0\omega_1+\alpha_2^0\omega_2,$$

即定理 2 成立.

另一方面, 不难验证$\overline{\boldsymbol{x}}=(0,0)$是问题(UP)的鲁棒弱有效解. 对于$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})=(0,0,1,1,0,0,0,0,1,0,0)$, 易得

$$x_1^4+x_1^2-3\in\Sigma_4^2,\quad \boldsymbol{A}_1^0\geq 0,\quad \boldsymbol{A}_2^0\geq 0,\quad \boldsymbol{B}_1^0\geq 0,$$

因此$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})$是问题(RD)的可行解. 又由定理 2 可知, 对任意的$(\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\lambda})\in\overline{C}$,

$$\omega_r\leqslant\widetilde{F}_r(\overline{\boldsymbol{x}})=0=\overline{\omega}_r,\qquad r=1,2.$$

故$(\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\lambda}})$是问题(RD)的弱有效解, 即定理 3 成立.

因为平方和条件可以等价地表示为线性矩阵不等式, 所以下面举例说明求解问题(RD)可以转化为求解相应的半定规划问题.

例 2 考虑例 1 中的 SOS 松弛对偶问题(RD):

$$\max_{\boldsymbol{\omega}\in\mathbb{R}^2,\,\boldsymbol{\alpha}\in\mathbb{R}_+^2\times\mathbb{R}^4,\,\boldsymbol{\lambda}\in\mathbb{R}_+\mathbb{R}^2}\{\omega_1,\omega_2\},$$

$$\text{s.t.}\ \sum_{r=1}^{2}\left(\alpha_r^0 b_r^0+\sum_{i=1}^{2}\alpha_r^i b_r^i\right)+\left(\lambda_1^0 q_1^0+\sum_{j=1}^{2}\lambda_1^j q_1^j\right)-\sum_{r=1}^{2}\alpha_r^0\omega_r\in\Sigma_4^2[\boldsymbol{x}],$$

$$\alpha_r^0\boldsymbol{A}_r^0+\sum_{i=1}^{2}\alpha_r^i\boldsymbol{A}_r^i\geq 0,\quad r=1,2,\quad \lambda_1^0\boldsymbol{B}_1^0+\sum_{j=1}^{2}\lambda_1^j\boldsymbol{B}_1^j\geq 0,\quad \sum_{r=1}^{2}\alpha_r^0=1.$$

设

$$\sigma:=\sum_{r=1}^{2}\left(\alpha_r^0 b_r^0+\sum_{i=1}^{2}\alpha_r^i b_r^i\right)+\left(\lambda_1^0 q_1^0+\sum_{j=1}^{2}\lambda_1^j q_1^j\right)-\sum_{r=1}^{2}\alpha_r^0\omega_r,$$

显然

$$\sigma(\boldsymbol{x})=\alpha_1^0x_1^4+\alpha_2^0x_1^2+(-2\alpha_1^0-\alpha_2^0+3\lambda_1^0+\lambda_1^2)x_2+\lambda_1^1x_1-\alpha_1^0+\alpha_1^1-2\alpha_2^0+\alpha_2^2-\alpha_1^0\omega_1-\alpha_2^0\omega_2.$$

由$\sigma\in\Sigma_4^2[\boldsymbol{x}]$和文献[24]中引理 3.33 可知, 存在对称矩阵$\boldsymbol{W}\in S^6$, 使得

$$\sigma=\boldsymbol{X}\boldsymbol{W}\boldsymbol{X}^{\mathrm{T}}\ \text{且}\ \boldsymbol{W}\geq 0,\tag{11}$$

其中$\boldsymbol{X}:=(1,x_1,x_2,x_1x_2,x_1^2,x_2^2)$. 设

$$\boldsymbol{W}=\begin{pmatrix}W_1 & W_2 & W_3 & W_4 & W_5 & W_6\\ W_2 & W_7 & W_8 & W_9 & W_{10} & W_{11}\\ W_3 & W_8 & W_{12} & W_{13} & W_{14} & W_{15}\\ W_4 & W_9 & W_{13} & W_{16} & W_{17} & W_{18}\\ W_5 & W_{10} & W_{14} & W_{17} & W_{19} & W_{20}\\ W_6 & W_{11} & W_{15} & W_{18} & W_{20} & W_{21}\end{pmatrix},$$

于是由式(11)可知,

$$W_{19}=\alpha_1^0,\quad 2W_3=-2\alpha_1^0-\alpha_2^0+3\lambda_1^0+\lambda_1^2,\quad 2W_2=\lambda_1^1,\quad W_1=-\alpha_1^0+\alpha_1^1-2\alpha_2^0+\alpha_2^2-\alpha_1^0\omega_1-\alpha_2^0\omega_2,$$

$$2W_5+W_7=\alpha_2^0,\quad 2W_4+2W_8=0,\quad W_{12}+2W_6=0,\quad 2W_9+2W_{14}=0,\quad 2W_{10}=0,$$

$$2W_{11}+2W_{13}=0,\quad 2W_{15}=0,\quad 2W_{17}=0,\quad 2W_{20}+W_{16}=0,\quad 2W_{18}=0,\quad W_{21}=0.$$

设 $W_5=\gamma\in\mathbb{R}$，$W_4=\tau\in\mathbb{R}$，$W_6=\varphi\in\mathbb{R}$，$W_9=\eta\in\mathbb{R}$，$W_{11}=\mu\in\mathbb{R}$，$W_{20}=\beta\in\mathbb{R}$，则

$$W_7=\alpha_2^0-2\gamma,\quad W_8=-\tau,\quad W_{12}=-2\varphi,\quad W_{14}=-\eta,\quad W_{13}=-\mu,\quad W_{16}=-2\beta.$$

因此，SOS 松弛对偶问题(RD)可转化为如下半定规划问题：

$$\max_{\substack{\boldsymbol{\omega}\in\mathbb{R}^2,\ \gamma\in\mathbb{R},\ \tau\in\mathbb{R},\ \varphi\in\mathbb{R},\ \eta\in\mathbb{R},\\ \mu\in\mathbb{R},\ \beta\in\mathbb{R},\ \boldsymbol{\alpha}\in\mathbb{R}_+^2\times\mathbb{R}^4,\ \boldsymbol{\lambda}\in\mathbb{R}_+\times\mathbb{R}^2}}\{\omega_1,\omega_2\},$$

$$\text{s.t.}\begin{pmatrix}
-\alpha_1^0+\alpha_1^1-2\alpha_2^0+\alpha_2^2-\alpha_1^0\omega_1-\alpha_2^0\omega_2 & \frac{\lambda_1^1}{2} & -\alpha_1^0-\frac{\alpha_2^0}{2}+\frac{3\lambda_1^0}{2}+\frac{\lambda_1^2}{2} & \tau & \gamma & \varphi\\
\frac{\lambda_1^1}{2} & \alpha_2^0-2\gamma & -\tau & \eta & 0 & \mu\\
-\alpha_1^0-\frac{\alpha_2^0}{2}+\frac{3\lambda_1^0}{2}+\frac{\lambda_1^2}{2} & -\tau & -2\varphi & -\mu & -\eta & 0\\
\tau & \eta & -\mu & -2\beta & 0 & 0\\
\gamma & 0 & -\eta & 0 & \alpha_1^0 & \beta\\
\varphi & \mu & 0 & 0 & \beta & 0
\end{pmatrix}\succeq 0,$$

$$\begin{pmatrix}\alpha_1^0 & 0 & \alpha_1^1\\ 0 & \alpha_1^0 & \alpha_1^2\\ \alpha_1^1 & \alpha_1^2 & \alpha_1^0\end{pmatrix}\succeq 0,\quad \begin{pmatrix}2\alpha_2^0 & 0 & \alpha_2^1\\ 0 & 4\alpha_2^0 & \alpha_2^2\\ \alpha_2^1 & \alpha_2^2 & \alpha_2^0\end{pmatrix}\succeq 0,\quad \begin{pmatrix}\lambda_1^0 & 0 & \lambda_1^1\\ 0 & \lambda_1^0 & \lambda_1^2\\ \lambda_1^1 & \lambda_1^2 & \lambda_1^0\end{pmatrix}\succeq 0,\quad \sum_{r=1}^{2}\alpha_r^0=1.$$

综上所述，本文主要对一类带不确定参数的 SOS 凸多项式优化问题进行了研究. 首先给出了一类不确定 SOS 凸多项式系统的鲁棒择一性定理；然后在不借助任何约束规范或正则性条件的前提下，利用不确定 SOS 多项式不等式系统的鲁棒择一性定理，研究了该不确定 SOS 凸多项式优化问题与其 SOS 松弛对偶问题之间的对偶性质；最后借助数值算例说明了对偶性结果的可行性.

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