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Uncertainty Principle for the Quaternion Linear Canonical Transform in Terms of Covariance

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Abstract: An uncertainty principle (UP), which offers information about a signal and its Fourier transform in the time-frequency plane, is particularly powerful in mathematics, physics and signal processing community. Under the polar coordinate form of quaternion-valued signals, the UP of the two-sided quaternion linear canonical transform (QLCT) is strengthened in terms of covariance. The condition giving rise to the equal relation of the derived result is obtained as well. The novel UP with covariance can be regarded as one in a tighter form related to the QLCT. It states that the product of spreads of a quaternion-valued signal in the spatial domain and the QLCT domain is bounded by a larger lower bound.

Keywords: uncertainty principle; quaternion linear canonical transform; quaternion-valued signals; covariance

1 Introduction

An uncertainty principle (UP), or time-bandwidth theorem in the time-frequency plane, plays a key role in various fields, such as the signal processing and quantum physics. It states that a signal cannot be simultaneously sharply located in both time and frequency domains. In other words, for a given unit energy signal f, the product of spreads of the signal in time domain and frequency domain is bounded by a lower bound [1]

$$\int_{\mathbb{R}}(t-t_0)^2|f(t)|^2\mathrm{d}t\int_{\mathbb{R}}(w-w_0)^2|F(w)|^2\mathrm{d}w\geqslant\frac{1}{4}\quad(1)$$

where

$$t_0=\int_{\mathbb{R}}t|f(t)|^2\mathrm{d}t,\ w_0=\int_{\mathbb{R}}w|F(w)|^2\mathrm{d}w\quad(2)$$

Herein, $F(w)$ denotes the classical Fourier transform (FT) of the signal $f(t)$. t_0 and w_0 stand for the mean in time and frequency domain, respectively. Since UPs do not depend on the location of means, let $t_0=w_0=0$ for the sake of simplicity. If the signal f is expressed in the polar form $f(t)=|f(t)|\,\mathrm{e}^{\mathrm{i}\theta(t)}$, a stronger UP is given by

$$\int_{\mathbb{R}}t^2|f(t)|^2\mathrm{d}t\int_{\mathbb{R}}w^2|F(w)|^2\mathrm{d}w\geqslant\frac{1}{4}+\mathrm{Cov}_{tw}^2\quad(3)$$

where Cov_{tw} is the covariance on time t and frequency w of f, and defined by

$$\mathrm{Cov}_{tw}=\int_{\mathbb{R}}t\theta'(t)|f(t)|^2\mathrm{d}t\quad(4)$$

Compared with (1), the lower bound of (3) is tighter, and in (4), $\theta'(t)$ is phase derivative related to the instantaneous frequency of f. Since the UP is of importance in various scientific fields such as optics, mathematics, and information theory, there are many efforts to extend (1) and (3) to various types of signals and integral transforms, see [2-4].

The linear canonical transform (LCT) is a three-free-parameter class of linear integral transform, which includes many important integral

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transforms as its special cases such as the FT, the fractional Fourier transform (FRFT), the Fresnel transform (FST), the Lorentz transform (LT), and other transforms [5]. It is an effective tool for non-stationary signal analysis and used widely in parameter estimation, sampling process for non-bandlimited signals with non-linear Fourier atoms [6]. Related works on LCT-based UPs can be referred to [3, 7, 8], which describe the constraint relationship between spreads of the time and frequency in the LCT domain.

Recently, the authors extend the LCT in the framework of quaternion algebra, called as quaternion linear canonical transform (QLCT) [9]. It provides an effective representation of the hypercomplex signal in the quaternion-valued frequency domain, which can be used for color image analysis. A natural question is what do the UPs in the QLCT setting obey? The UP in (1) has been generalized for the QLCT in the name of component-wise UP, focusing on lower bounds in the spatial and frequency domain [10]. Zhang et al. derived extension of component-wise UP, named as directional UP and modified UP using the relationship between QLCT and quaternion Fourier transform (QFT) [11]. Furthermore, Zhu et al. generalized component-wise UP in two different QLCT domains, and proposed other kinds of UPs, i.e., local UP, Benedicks-Amrein-Berthier UP and entropic UP with applications in signal recovery and physical quantum [12]. However, the QLCT-based UP similar to (3) with a covariance form has not derived yet. It is therefore interesting and worthwhile to investigate these kinds of UPs and obtain much more valuable results associated with the QLCT.

The rest of this paper is organized as follows. Section 2 provides a brief introduction to some general definitions of quaternion algebra and the QLCT. New UPs in terms of covariance related to the QLCT are given in Section 3. Finally, Section 4 concludes this paper.

2 Preliminaries

In this present section, some basic facts on the quaternion algebra and the QLCT are reviewed.

2.1 Quaternion Algebra

The quaternion algebra is an extension of the complex number to 4D algebra. It was first invented by W. R. Hamilton in 1843 and classically denoted by $\mathbb{H}$ in his honor. For every element of $\mathbb{H}$, it has a Cartesian form given by

$$\mathbb{H} = \{q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} | q_0, q_1, q_2, q_3 \in \mathbb{R}\} \quad (5)$$

where i, j, k are imaginary units obeying Hamilton's multiplication rules

$$\begin{aligned} &\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \\ &\mathrm{jk} = -\mathrm{kj} = \mathrm{i},\ \mathrm{ki} = -\mathrm{ik} = \mathrm{j} \end{aligned} \quad (6)$$

For every quaternionic number $q \in \mathbb{H}$, it can be decomposed into the scalar part and the non-scalar part of q, which is denoted as $\mathrm{Sc}(q)$=q_0, and $\mathrm{NSc}(q) = q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$, respectively.

Every quaternion $q \in \mathbb{H}$ has a quaternion conjugate $\bar{q}$ given by

$$\bar{q} = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k} = \mathrm{Sc}(q) - \mathrm{NSc}(q) \quad (7)$$

which is an anti-involution, i.e.,

$$\overline{qp} = \bar{p}\bar{q} \quad (8)$$

According to (8), the norm or modulus of $q \in \mathbb{H}$ is defined by

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \quad (9)$$

and the inverse of $q \in \mathbb{H} \setminus \{0\}$ is $q^{-1} = \frac{\bar{q}}{|q|^2}$. Any quaternion q can also be written as a polar form, i.e., $q = |q|\,\mathrm{e}^{\mu\theta}$, where $\mathrm{e}^{\mu\theta} = \cos\theta + \mu\sin\theta$ is in accordance with Euler's formula. Herein, $\mu = \frac{\mathrm{Nsc}(q)}{|\mathrm{Nsc}(q)|}$ is the axis verifying $\mu^2 = -1$, and $\theta = \arctan\frac{|\mathrm{Nsc}(q)|}{\mathrm{Sc}(q)}$ is the phase of q, respectively.

A quaternion-valued signal $f : \mathbb{R}^2 \to \mathbb{H}$ can be expressed as

$$f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + f_1(\boldsymbol{x})\mathrm{i} + f_2(\boldsymbol{x})\mathrm{j} + f_3(\boldsymbol{x})\mathrm{k} \quad (10)$$

where $f_l : \mathbb{R}^2 \to \mathbb{R}, l = 0, 1, 2, 3$ and $\boldsymbol{x} = (x_1, x_2)$. A color image can be considered as a pure quaternion-valued signal, i.e.,

$$f(\boldsymbol{x}) = f_{\mathrm{R}}(\boldsymbol{x})\mathrm{i} + f_{\mathrm{G}}(\boldsymbol{x})\mathrm{j} + f_{\mathrm{B}}(\boldsymbol{x})\mathrm{k} \tag{11}$$

where f_{R}, f_{G}, f_{B} represents the red, green, blue element of the color image, respectively.

Quaternion value inner product of two signals $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\mathrm{d}x_2 \tag{12}$$

The associated norm of $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined as

$$\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} = \langle f, f\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} < \infty \tag{13}$$

Lemma 1 If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then the Cauchy-Schwarz inequality holds [13]

$$\left|\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}\right|^2 \leqslant \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|g\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{14}$$

If and only if $f = -\lambda g$ for some quaternionic parameter $\lambda \in \mathbb{H}$, the equality holds.

2.2 The Quaternion Linear Canonical Transform

The QLCT is firstly defined by Kou, et al., which is a generalization of the LCT in the frame of quaternion algebra [14]. Due to the non-commutativity of quaternion multiplication, there are three different types of the QLCT: the left-sided QLCT, the right-sided QLCT, and the two-sided QLCT. In this paper, we mainly focus on the two-sided QLCT.

Definition 1 Let $\boldsymbol{A}_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $\det(\boldsymbol{A}_i) = 1$, for $i = 1, 2$. The two-sided QLCT of signals $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is defined by [15]

$$\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u}) = \int_{\mathbb{R}^2} K^{\mathrm{i}}_{\boldsymbol{A}_1}(x_1, u_1) f(\boldsymbol{x}) K^{\mathrm{j}}_{\boldsymbol{A}_2}(x_2, u_2)\mathrm{d}\boldsymbol{x} \tag{15}$$

where $\boldsymbol{u} = (u_1, u_2)$ with the kernel signals $K^{\mathrm{i}}_{\boldsymbol{A}_1}(x_1, u_1)$ and $K^{\mathrm{j}}_{\boldsymbol{A}_2}(x_2, u_2)$ respectively, given by

$$K^{\mathrm{i}}_{\boldsymbol{A}_1}(x_1, u_1) = \begin{cases} \dfrac{1}{\sqrt{2\pi b_1}} \mathrm{e}^{\mathrm{i}\left(\frac{a_1}{2b_1}x_1^2 - \frac{x_1u_1}{b_1} + \frac{d_1}{2b_1}u_1^2 - \frac{\pi}{4}\right)}, & b_1 \neq 0 \\ \sqrt{d_1}\mathrm{e}^{\mathrm{i}\frac{c_1d_1}{2}u_1^2}, & b_1 = 0 \end{cases} \tag{16}$$

and

$$K^{\mathrm{j}}_{\boldsymbol{A}_2}(x_2, u_2) = \begin{cases} \dfrac{1}{\sqrt{2\pi b_2}} \mathrm{e}^{\mathrm{j}\left(\frac{a_2}{2b_2}x_2^2 - \frac{x_2u_2}{b_2} + \frac{d_2}{2b_2}u_2^2 - \frac{\pi}{4}\right)}, & b_2 \neq 0 \\ \sqrt{d_2}\mathrm{e}^{\mathrm{j}\frac{c_2d_2}{2}u_2^2}, & b_2 = 0 \end{cases} \tag{17}$$

From above definition, it is noted that for $b_i = 0, i = 1, 2$ the QLCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b_i \neq 0$ in the following section unless stated otherwise. Under some suitable conditions, the QLCT above is invertible and the inversion is given in **Definition 2.**

Definition 2 Suppose $f \in L^1(\mathbb{R}^2, \mathbb{H})$, then the inversion of the QLCT of f is given by [14]

$$f(\boldsymbol{x}) = \{\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\}^{-1}\{f\}(\boldsymbol{x}) = \int_{\mathbb{R}^2} K^{-\mathrm{i}}_{\boldsymbol{A}_1}(x_1, u_1)\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})K^{-\mathrm{j}}_{\boldsymbol{A}_2}(x_2, u_2)\mathrm{d}\boldsymbol{u} \tag{18}$$

Since it is impossible to establish the Plancherel identity of the two-sided QLCT using the inner product in (12), a Q-module is defined in the following [10, 16].

Definition 3 Suppose $f \in L^2(\mathbb{R}^2, \mathbb{H})$, the Q-module of the QLCT, $\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})$, is defined by [10, 16]

$$\left|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right|_Q = \sqrt{\sum_{m=0}^{3}\left|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f_m\}(\boldsymbol{u})\right|^2} \tag{19}$$

Furthermore, the L^p-norm of $\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})$ is defined by

$$\left\|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right\|_{Q,p} = \left(\int_{\mathbb{R}^2}\left|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right|^p_Q \mathrm{d}\boldsymbol{u}\right)^{1/p} < \infty \tag{20}$$

Some elementary properties of the QLCT used in the sequel is listed in **Lemma 2** and **Lemma 3**.

Lemma 2 Every quaternion-valued signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and its QLCT are related to the Plancherel identity in the following way [10]

$$\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} = \left\|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{u})\right\|^2_{Q,2} \tag{21}$$

Lemma 3 Set $\boldsymbol{A}_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter, $b_i \neq 0$ such that $\det(\boldsymbol{A}_i) = 1$ for $i = 1, 2$. If $f \in L^2(\mathbb{R}^2, \mathbb{H})$, $\dfrac{\partial f}{\partial x_i}$ exists and is in $L^1(\mathbb{R}^2, \mathbb{H})$, then [10]

$$\left\|u_1\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right\|^2_{Q,2}= \left\|b_1\frac{\partial}{\partial x_1}f(\boldsymbol{x})+\mathrm{i}a_1x_1f(\boldsymbol{x})\right\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{22}$$

and

$$\left\|u_2\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right\|^2_{Q,2}= \left\|b_2\frac{\partial}{\partial x_2}f(\boldsymbol{x})+a_2f(\boldsymbol{x})x_2\mathrm{j}\right\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{23}$$

Many other important properties of the QLCT have been derived, such as shift, modulation, and asymptotic properties [14, 15]. Other kinds of UPs in the QLCT domain have been studied [9, 11]. Applications of the QLCT to study the generalized swept-frequency filters, and color image analysis have been presented [17].

3 Main Results

The main result is the following. **Theorem 1** presents an UP associated with the QLCT in the covariance form.

Theorem 1 Let $f(\boldsymbol{x})=|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\in L^2(\mathbb{R}^2,\mathbb{H})$, and $\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{u})$ is the QLCT of f with the matrix parameter $\boldsymbol{A}_i, i=1,2$. If x_kf, $\dfrac{\partial}{\partial x_k}f$, $u_k\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}\in L^2(\mathbb{R}^2,\mathbb{H})$ for $k=1,2$, then

$$\Delta x_1^2\Delta u_1^2\geqslant\frac{b_1^2}{4}+|b_1\mathrm{Cov}_{x_1w_1}+\mathrm{i}a_1\Delta x_1^2|^2 \tag{24}$$

and

$$\Delta x_2^2\Delta u_2^2\geqslant\frac{b_2^2}{4}+|b_2\mathrm{Cov}_{x_2w_2}+a_2J|^2 \tag{25}$$

where

$$\begin{aligned}\Delta x_k^2&=\int_{\mathbb{R}^2}x_k^2|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\\ \Delta u_k^2&=\int_{\mathbb{R}^2}u_k^2|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})|^2_Q\mathrm{d}\boldsymbol{x}\end{aligned} \tag{26}$$

and $J=\int_{\mathbb{R}^2}x_2^2f(\boldsymbol{x})\mathrm{j}\overline{f(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\in\mathrm{NSc}(q)$. The covariance of x_k and w_k is

$$\mathrm{Cov}_{x_kw_k}=\int_{\mathbb{R}^2}\mathrm{NSc}\left[\left(\frac{\partial}{\partial x_k}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right]|f(\boldsymbol{x})|^2x_k\mathrm{d}\boldsymbol{x} \tag{27}$$

The (24) and (25) holds if and only if $f(\boldsymbol{x})=\mathrm{e}^{-\frac{\alpha_1}{2b_1}x_1^2-\frac{\alpha_2}{2b_2}x_2^2}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}$ which satisfies

$$\begin{cases}\left(\dfrac{\partial}{\partial x_1}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}=\dfrac{\boldsymbol{\beta}_1-\mathrm{i}a_1}{b_1}x_1\\ \left(\dfrac{\partial}{\partial x_2}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}=\\ \dfrac{\boldsymbol{\beta}_2}{b_2}x_2-\dfrac{a_2x_2}{b_2}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\mathrm{j}\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\end{cases} \tag{28}$$

Herein, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are non-scalar part of quaternions.

Proof For $k=1$, using the derivation property of QLCT in **Lemma** 3, it's obtained that

$$\begin{aligned}\Delta x_1^2\Delta u_1^2=&b_1^2\int_{\mathbb{R}^2}x_1^2|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_1}f(\boldsymbol{x})+\mathrm{i}\frac{a_1}{b_1}x_1f(\boldsymbol{x})\right|^2\mathrm{d}\boldsymbol{x}\geqslant\\ &\left|\int_{\mathbb{R}^2}\left(b_1\frac{\partial}{\partial x_1}f(\boldsymbol{x})+\mathrm{i}a_1x_1f(\boldsymbol{x})\right)x_1\overline{f(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\right|^2=\\ &\left|\int_{\mathbb{R}^2}b_1\left(\frac{\partial}{\partial x_1}f(\boldsymbol{x})\right)x_1\overline{f(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}+\mathrm{i}a_1\Delta x_1^2\right|^2\end{aligned}$$

For $f(\boldsymbol{x})=|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}$, the following is derived

$$\int_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_k}f(\boldsymbol{x})\right)\overline{f(\boldsymbol{x})}x_k\mathrm{d}\boldsymbol{x}=-\frac{1}{2}+\mathrm{Cov}_{x_kw_k} \tag{29}$$

where $\mathrm{Cov}_{x_kw_k}$ is the covariance between x_k and w_k [13]. Then,

$$\begin{aligned}\Delta x_1^2\Delta u_1^2\geqslant&\left|b_1\left(-\frac{1}{2}+\mathrm{Cov}_{x_1w_1}\right)+\mathrm{i}a_1\Delta x_1^2\right|^2=\\ &\frac{b_1^2}{4}+|b_1\mathrm{Cov}_{x_1w_1}+\mathrm{i}a_1\Delta x_1^2|^2\end{aligned}$$

For $k=2$, in a similar way, the following is derived

$$\begin{aligned}\Delta x_2^2\Delta u_2^2=&b_2^2\int_{\mathbb{R}^2}x_2^2|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_2}f(\boldsymbol{x})+\frac{a_2}{b_2}f(\boldsymbol{x})x_2\mathrm{j}\right|^2\mathrm{d}\boldsymbol{x}\geqslant\\ &\left|\int_{\mathbb{R}^2}\left(b_2\frac{\partial}{\partial x_2}f(\boldsymbol{x})+a_2f(\boldsymbol{x})x_2\mathrm{j}\right)x_2\overline{f(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\right|^2=\\ &\frac{b_2^2}{4}+|b_2\mathrm{Cov}_{x_2w_2}+a_2J|^2\end{aligned}$$

with $J=\int_{\mathbb{R}^2}x_2^2f(\boldsymbol{x})\mathrm{j}\overline{f(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\in\mathrm{NSc}(q)$.

The condition of equality in (24) holds if $b_1\dfrac{\partial}{\partial x_1}f(\boldsymbol{x})+\mathrm{i}a_1x_1f(\boldsymbol{x})=q_1x_1f(\boldsymbol{x})$, where $q_1=\pm\alpha_1+\boldsymbol{\beta}_1,\alpha_1>0,\boldsymbol{\beta}_1=\mathrm{Nsc}(q_1)$. Putting it into the polar form of $f(\boldsymbol{x})=|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}$, we derive

$$b_1\frac{\partial}{\partial x_1}f(\boldsymbol{x})+\mathrm{i}a_1x_1f(\boldsymbol{x})=$$
$$b_1\left[\left(\frac{\partial}{\partial x_1}|f(\boldsymbol{x})|\right)\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}+|f(\boldsymbol{x})|\left(\frac{\partial}{\partial x_1}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\right]+$$
$$\mathrm{i}a_1x_1|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}=(\pm\alpha_1+\boldsymbol{\beta}_1)x_1|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}$$

Therefore, the condition of equation is

$$\begin{cases}b_1\dfrac{\partial}{\partial x_1}|f(\boldsymbol{x})|=\pm\alpha_1x_1|f(\boldsymbol{x})|\\ \left(\dfrac{\partial}{\partial x_1}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}=\dfrac{\boldsymbol{\beta}_1-\mathrm{i}a_1}{b_1}x_1\end{cases}\tag{30}$$

That is to say, $|f(\boldsymbol{x})|=\mathrm{e}^{\pm\frac{\alpha_1}{2b_1}x_1^2}$ with $\left(\dfrac{\partial}{\partial x_1}\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}\right)\times\mathrm{e}^{-\mu(\boldsymbol{x})\theta(\boldsymbol{x})}=\dfrac{\boldsymbol{\beta}_1-\mathrm{i}a_1}{b_1}x_1$. The same method is used when $k=2$. This completes the proof.

Remark 1 Now choosing the special matrix parameter $\boldsymbol{A}_i=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\in\mathbb{R}^{2\times2}$, the QLCT of f is reduced to the QFT $\mathcal{F}\{f\}$ of quaternion-valued f with multiplicator factors as follows [9, 11, 18]

$$\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})=$$
$$\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\left[\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}x_1w_1}f(\boldsymbol{x})\mathrm{e}^{-\mathrm{j}x_2w_2}\mathrm{d}\boldsymbol{x}\right]\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}=$$
$$\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\mathcal{F}\{f\}(\boldsymbol{w})\mathrm{e}^{-j\frac{\pi}{4}}\tag{31}$$

Then, UP in terms of covariance related to the two-sided QFT is derived

$$\int_{\mathbb{R}^2}x_k^2|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^2}w_k^2|\mathcal{F}\{f\}(\boldsymbol{w})|_Q^2\mathrm{d}\boldsymbol{w}\geqslant\frac{1}{4}+\left|\mathrm{Cov}_{x_kw_k}\right|^2\tag{32}$$

Remark 2 Since $\dfrac{b_1^2}{4}+|b_1\mathrm{Cov}_{x_1w_1}+\mathrm{i}a_1\Delta x_1^2|^2\geqslant\dfrac{b_1^2}{4}$ and $\dfrac{b_2^2}{4}+|b_2\mathrm{Cov}_{x_2w_2}+a_2J|^2\geqslant\dfrac{b_2^2}{4}$, the lower bounder in **Theorem 1** is much stronger than that in literature [10]. When estimating the effective bandwidth in optical systems, time-frequency analysis and affine modulation schemes, the new derived UP in this paper could derive a tighter spread in the QLCT domain. In the following, a directional QLCT-based UP is derived as a corollary of **Theorem 1**.

Corollary 1 Let $f(\boldsymbol{x})=|f(\boldsymbol{x})|\mathrm{e}^{\mu(\boldsymbol{x})\theta(\boldsymbol{x})}$. If $x_kf,\dfrac{\partial}{\partial x_k}f,u_k\mathcal{L}^{\mathbb{H}}_{A_1,A_2}f\in L^2(\mathbb{R}^2,\mathbb{H})$ for $k=1,2$, then

$$\Delta\boldsymbol{x}^2\Delta\boldsymbol{u}^2\geqslant\left[\left(\frac{b_1^2}{4}+\left|b_1\mathrm{Cov}_{x_kw_k}+\mathrm{i}a_1\Delta x_1^2\right|^2\right)^{1/2}+\left(\frac{b_2^2}{4}+\left|b_2\mathrm{Cov}_{x_kw_k}+a_2J\right|^2\right)^{1/2}\right]^2\tag{33}$$

where

$$\Delta\boldsymbol{x}^2=\int_{\mathbb{R}^2}(x_1^2+x_2^2)|f(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x},$$
$$\Delta\boldsymbol{u}^2=\int_{\mathbb{R}^2}(u_1^2+u_2^2)\left|\mathcal{L}^{\mathbb{H}}_{\boldsymbol{A}_1,\boldsymbol{A}_2}\{f\}(\boldsymbol{u})\right|_Q^2\mathrm{d}\boldsymbol{u}\tag{34}$$

is the directional spread of a signal in the time and the QLCT domain, respectively. According to the Cauchy-Schwarz inequality of discrete form, it is easy to derive the final result.

4 Conclusion

In this paper, based on the basic properties of the two-sided QLCT, a tighter UP in terms of the covariance is proposed for a quaternion-valued signal in polar coordinate form. The new derived result describes a stronger lower bound of spreads of a quaternion-valued signal in spatial domain and the QLCT domain, which includes those of UPs related to the QFT as its special case. It offers a method to estimate the effective bandwidth in the QLCT settings and plays a fundamental role in the further studies on the time-frequency analysis, sampling theory in 2D case, and filter design in the quaternion setting. Besides, the discrete UPs relate closely to the problem of signal recovery; novel derived UPs in the QLCT domains can further contribute to solving the problem of quaternion-valued signal recovery in the practical applications. In the forthcoming work, the following work will be focused on: 1) tighter UPs for left-sided QLCT and right-sided QLCT; 2) tighter UPs in arbitrary two different QLCT domains; 3) UPs for discrete signal and sparse recovery in quaternion setting.

References:

[1] P. Dang, G. T. Deng, and T. Qian,"A tighter uncertainty principle for linear canonical transform in terms of phase derivative," *IEEE Transaction on Signal Processing*, vol. 61, no. 21, pp. 5153-5164,

2013.

[2] G. L. Xu, X. T. Wang, and X. G. Xu,"On uncertainty principle for the linear canonical transform of complex signals," *IEEE Transaction on Signal Processing*, vol. 58, no. 9, pp. 4916-4918, 2010.

[3] K. K. Sharma and S. D. Joshi,"Uncertainty principle for real signals in the linear canonical transform domains," *IEEE Transaction on Signal Processing*, vol. 56, no. 7, pp. 2677-2683, 2008.

[4] J. Zhao, R. Tao, and Y. Wang,"On signal moments and uncertainty relations associated with linear canonical transform," *Signal Processing*, vol. 90, no. 9, pp. 2686-2689, 2010.

[5] D. Y. Wei and H. M. Hu,"Sparse discrete linear canonical transform and its applications," *Signal Processing*, vol. 183, pp. 108046, 2021.

[6] Z. C. Zhang,"Jittered sampling in linear canonical domain," *IEEE Communication Letters*, vol. 24, no. 7, pp. 1529-1533, 2020.

[7] W. B. Gao and B. Z. Li,"Uncertainty principles for the short-time linear canonical transform of complex signals," *Digital Signal Processing*, vol. 111, pp. 102953, 2021.

[8] Z. C. Zhang,"Uncertainty principle for linear canonical transform using matrix decomposition of absolute spread matrix," *Digital Signal Processing*, vol. 89, pp. 145-154, 2019.

[9] X. Zhu and S. Zheng,"Uncertainty principles for the two-sided quaternion linear canonical transform," *Circuits Systems and Signal Processing*, vol. 39, no. 9, pp. 4436-4458, 2020.

[10] K. I. Kou, J. Ou, and J. Morais,"Uncertainty principles associated with quaternionic linear canonical transforms," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 10, pp. 2722-2736, 2016.

[11] Y. N. Zhang and B. Z. Li,"Novel uncertainty principles for two-sided quaternion linear canonical transform," *Advances in Applied Clifford Algebras*, vol. 28, no. 1, pp. 1-15, 2018.

[12] X. Y. Zhu and S. Z. Zheng,"On uncertainty principle for the two-sided quaternion linear canonical transform," *Journal of Pseudo-Differential Operators and Applications*, vol. 12, no. 1, pp. 1-25, 2021.

[13] Y. Yang and K. I. Kou,"Novel uncertainty principles associated with 2D quaternion Fourier transforms," *Integral Transforms and Special Functions*, vol. 27, no. 3, pp. 213-226, 2016.

[14] K. I. Kou, J. Y. Ou, and J. Morais,"On Uncertainty principle for quaternionic linear canonical transform," *Abstract and Applied Analysis*, vol. 2013, no. 1, pp. 94-121, 2013.

[15] K. I. Kou and J. Morais,"Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem," *Applied Mathematics and Computation*, vol. 247, pp. 675-688, 2014.

[16] L. P. Chen, K. I. Kou, and M. S. Liu,"Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform," *Journal of Mathematical Analysis and Applications*, vol. 423, no. 1, pp. 681-700, 2015.

[17] M. Bahri,"Quaternion linear canonical transform application," *Global Journal of Pure and Applied Mathematics*, vol. 11, no. 1, pp. 19-24, 2015.

[18] M. Bahri, E. S. M. Hitzer, A. Hayashi, and R. Ashino,"An uncertainty principle for quaternion Fourier transform," *Computers and Mathematics with Applications*, vol. 56, no. 9, pp. 2398-2410, 2008.

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