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具有非局部结构阻尼的非自治梁方程的时间依赖拉回吸引子

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摘要：利用时间依赖空间上的过程理论，研究具有非局部结构阻尼的非自治梁方程解的长时间动力学行为. 首先，用 Faedo-Galerkin 逼近方法得到解的适定性；其次，利用能量估计得到该动力系统在相应解空间中拉回吸收集的存在性；最后，用共圈技术和收缩函数的方法证明时间依赖拉回吸引子的存在性.

关键词：结构阻尼；收缩函数；时间依赖拉回吸引子；梁方程
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Time-Dependent Pullback Attractors for Non-autonomous Beam Equation with Nonlocal Structural Damping

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Abstract: We studied the long-time dynamic behavior of solution to non-autonomous beam equation with nonlocal structural damping by using the process theory in the time-dependent space. Firstly, we obtained the well-posedness of solution by using Faedo-Galerkin approximation method. Secondly, the existence of pullback absorption set of the dynamical system in the corresponding solution space was obtained by using energy estimation. Finally, we proved the existence of time-dependent pullback attractors by using the cocyclic technique and contraction function method.

Keywords: structural damping; contraction function; time-dependent pullback attractor; beam equation

0 引 言

令 Ω 是 $\mathbb{R}^N(N=5)$ 中具有光滑边界的有界域，本文考虑如下具有非局部结构阻尼的非自治梁方程：

$$\begin{cases} \varepsilon(t)u_{tt}+\Delta^2 u+N(\|\nabla u\|^2)(-\Delta)^{\theta}u_t+f(u)=g(x,t), & x\in\Omega,\quad t\geqslant\tau,\\ u(x,t)=0, & x\in\partial\Omega,\quad t\geqslant\tau,\\ u(x,\tau)=u_0(x),\quad u_t(x,\tau)=u_1(x), & x\in\Omega, \end{cases} \tag{1}$$

其中 $\theta\in\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$，$g(x,t)\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$ 是外力项.

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Woinowsky-Krieger[1]提出了梁方程；Ball[2-3]讨论了弹性梁方程的稳定性，并给出了该类方程严格的数学分析. 当问题(1)中的$\varepsilon(t)$恒为C时，关于这类问题的研究已有很多成果[4-8]. 但当$\varepsilon(t)$为正的单调递减函数时，方程(1)变得更复杂. 因为即使外力项$g(\cdot)$与时间t无关，对问题(1)这个动力系统仍需在非自治情形下进行讨论. 为解决此类问题，Conti等[9]基于拉回吸引性的最小性，提出了拉回吸引子的概念；刘亭亭等[10]基于Plate方程在弱拓扑空间中时间依赖全局吸引子的存在性和正则性，讨论了Plate方程在强拓扑空间中时间依赖拉回吸引子的存在性；汪璇等[11]研究了Kirchhoff型波方程时间依赖吸引子的存在性；Yang等[12]讨论了Kirchhoff型波方程拉回吸引子的存在性及其上半连续性. 当问题(1)中的外力项$g(\cdot)$与时间t有关时，其时间依赖动力学行为的研究目前文献报道较少. 由于问题(1)中的非局部结构阻尼项和非自治外力项会给该问题的研究带来本质性困难，因此，本文利用收缩函数及动力系统θ诱导出的共圈函数χ证明问题(1)时间依赖拉回吸引子的存在性.

1 预备知识

首先引入符号：$L^p=L^p(\Omega)$，$H^k=H^k(\Omega)$，$V_1=H_0^1(\Omega)$，$V_2=H_0^1\cap H^2$，$\|\cdot\|=\|\cdot\|_{L^2}$，$\|\cdot\|_p=\|\cdot\|_{L^p}$，其中$p\geqslant 2$，则有嵌入$V_2\hookrightarrow\hookrightarrow L^2\hookrightarrow V_{-2}$. 定义算子$A$：$V_2\to V_{-2}$，

$$\langle Au,v\rangle=\langle\Delta u,\Delta v\rangle,\qquad \forall\, u,v\in V_2,$$

则A是L^2上的自伴算子且在V_2上严格正，也可以定义A的幂A^s. Hilbert空间$V_s=D(A^{s/4})$具有以下内积与范数：

$$\langle u,v\rangle_{V_s}=\langle A^{s/4}u,A^{s/4}v\rangle,$$
$$\|u\|_{V_s}=\|A^{s/4}u\|.$$

则问题(1)可以改写成以下抽象的形式：

$$\begin{cases}\varepsilon(t)\,u_{tt}+Au+N(\|u\|_{V_1}^2)A^{\theta/2}u_t+f(u)=g(x,t), & x\in\Omega,\quad t\geqslant\tau,\\ u(\tau)=u_0(x),\quad u_t(\tau)=u_1(x), & x\in\Omega.\end{cases}\tag{2}$$

定义C是一个正常数，每个C在不同的不等式中表示不同的常数值，假设函数$\varepsilon(t)$，$N(\cdot)$，$f(u)$和$g(x,t)$满足下列条件：

(H_1) 函数$\varepsilon(t)\in C^1(\mathbb{R})$是一个递减有界函数，且满足

$$\lim_{t\to+\infty}\varepsilon(t)=0,\tag{3}$$

存在常数$L>0$，使得

$$\sup_{t\in\mathbb{R}}[\,|\varepsilon(t)|+|\varepsilon'(t)|\,]\leqslant L;\tag{4}$$

(H_2) $N(s)\in C^1(\mathbb{R}^+)$，$N(s)\geqslant N_R>0$，$s\in\mathbb{R}^+$；

(H_3) 对于$f\in C^2(\mathbb{R})$，$s\in\mathbb{R}$，满足

$$u_f=\liminf_{|s|\to\infty}\frac{f(s)}{s}>-\lambda_1,\tag{5}$$

其中$\lambda_1>0$为算子A的第一特征值，且当$N=5$时，有

$$|f''(s)|\leqslant C(1+|s|^{p-2}),\qquad 2\leqslant p<p_\theta=\frac{N+4\theta}{N-4}=5+4\theta;\tag{6}$$

(H_4) 外力项$g(x,t)\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$，且有

$$\int_{-\infty}^{t}\mathrm{e}^{\sigma s}\|g(s)\|^2\mathrm{d}s<+\infty,\tag{7}$$

其中$0<\sigma<\dfrac{\delta}{2}$，$\delta$为方程(1)做先验估计时所用实验函数中的常数系数.

注1 由式(5)和式(6)可知，存在常数η，使得当$0<\lambda_1-\eta\ll 1$时，有

$$\int_\Omega F(u)\mathrm{d}x\geqslant-\frac{\eta}{2}\|u\|^2-C\geqslant-\frac{\eta}{2\lambda_1}\|u\|_{V_2}^2-C,\tag{8}$$

$$\langle f(u),u\rangle\geqslant\int_\Omega F(u)\mathrm{d}x-\frac{\eta}{2\lambda_1}\|u\|_{V_2}^2-C,\tag{9}$$

其中 $F(u)=\int_0^u f(r)\mathrm{d}r$.

定义时间依赖空间为

$$\mathscr{H}_t=V_2\times L^2,$$

其对应的范数为

$$\|(u,u_t)\|_{\mathscr{H}_t}^2=\|u\|_{V_2}^2+\epsilon(t)\|u_t\|^2.$$

设(X,d)是完备的度量空间，(Q,ρ)为度量空间，也称为符号空间，$\theta=\{\theta_t\}_{t\in\mathbb{R}}$是定义在$Q$上的动力系统，且满足：

1) $\forall q\in Q$, $\theta_0(q)=q$;

2) $\forall q\in Q$, $t,\tau\in\mathbb{R}$, $\theta_{t+\tau}(q)=\theta_t(\theta_\tau(q))$;

3) $(t,q)\to\theta_t(q)$连续.

如果映射$\chi:\mathbb{R}^+\times Q\times X\to X$满足下列条件：

1) $\forall(q,x)\in Q\times X$, $\chi(0,q,x)=x$;

2) $\forall s,t\in\mathbb{R}^+$, $(q,x)\in Q\times X$, $\chi(t+s,q,x)=\chi(s,\theta_t(q),\chi(t,q,x))$.

则称映射$\chi:\mathbb{R}^+\times Q\times X\to X$是由$\theta$诱导出的共圈函数.

定义 1[10]　设X为Banach空间，B为X中的有界集，$\Phi(\cdot,\cdot)$是定义在$X\times X$上的函数，如果对任意的序列$\{x_n\}_{n=1}^\infty\subset B$，存在子列$\{x_{n_k}\}_{k=1}^\infty\subset\{x_n\}_{n=1}^\infty$，使得

$$\lim_{l\to\infty}\lim_{k\to\infty}\Phi(x_{n_k},x_{n_l})=0,$$

则称$\Phi(\cdot,\cdot)$为$X\times X$上的收缩函数，$\mathscr{C}$表示定义在$X\times X$上收缩函数的集合.

定理 1[10,12]　令(θ,χ)为$Q\times X$上的非自治动力系统，假设集合族$D=\{D_q\}_{q\in Q}$为χ的拉回吸收集，且χ是拉回D-渐近紧的，则χ拥有拉回吸引子$\mathscr{A}=\{A_q\}_{q\in Q}$，其中

$$A_q=\bigcap_{t\geqslant 0}\overline{\bigcup_{s\geqslant t}\chi(s,\theta_{-s}(q),D_{\theta_{-s}(q)})},\qquad q\in Q.\tag{10}$$

定理 2[10]　令(θ,χ)为$Q\times X$上的非自治动力系统，有界集族$\mathscr{D}=\{D_q\}_{q\in Q}$和$\widetilde{\mathscr{D}}=\{\widetilde{D}_q\}_{q\in Q}$满足对任意的$q\in Q$，存在一个$t_q=t(q,\mathscr{D},\widetilde{\mathscr{D}})\geqslant 0$，使得

$$\chi(t,\theta_{-t}(q),D_{\theta_{-t}(q)})\subset\widetilde{D}_q,\qquad\forall t\geqslant t_q.\tag{11}$$

假设对任意的$\epsilon>0$，$q\in Q$，存在一个$t=t(\epsilon,\widetilde{\mathscr{D}},q)\geqslant 0$及定义在$\widetilde{D}_{\theta_{-t(q)}}\times\widetilde{D}_{\theta_{-t(q)}}$上的收缩函数$\Phi_{t,q}(\cdot,\cdot)$，使得对任意的$x,y\in\widetilde{D}_{\theta_{-t(q)}}$，有

$$\|\chi(t,\theta_{-t}(q),x)-\chi(t,\theta_{-t}(q),y)\|_X\leqslant\epsilon+\Phi_{t,q}(x,y),$$

则称χ在X上是拉回$\mathscr{D}$-渐近紧的.

2　主要结果

2.1　解的存在唯一性

方程(2)解的存在性可通过标准的Faedo-Galerkin方法得到.

定理 3　假设条件$(H_1)\sim(H_4)$成立，$y_0\in\mathscr{H}_\tau$，$g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$，则在区间$[\tau,t]$上，对任意的初值$(u_0,v_0)$，问题(2)存在唯一解

$$u\in C([\tau,t],V_2(\Omega)),\qquad u_t\in C([\tau,t],L^2(\Omega)).$$

记

$$y_0=y_\tau=\{u_0,u_1\},\qquad y(t)=\{u(t),u_t(t)\}.$$

由问题(2)，可在空间$\mathscr{H}_t$中构造一个非自治动力系统，对任意的$t,\tau\in\mathbb{R}$，令$Q=\mathbb{R}$，$\theta_t(\tau)=\tau+t$，并定义

$$\chi(t,\tau,y_0)=y(t+\tau,\tau,y_0)=\{u(t+\tau),u_t(t+\tau)\},\quad\tau\in\mathbb{R},\quad s,t\geqslant 0,\quad y_0\in\mathscr{H}_\tau.\tag{12}$$

由方程(1)解的存在唯一性可知

$$\chi(t+s,\tau,y_0)=\chi(t,s+\tau,\chi(s,\tau,y_0)),\quad\tau\in\mathbb{R},\quad s,t\geqslant 0,\quad y_0\in\mathscr{H}_\tau,$$

且对任意的$\tau\in\mathbb{R}$，$t\geqslant 0$，则由式(12)定义的映射$\chi(t,\tau,\cdot):\mathscr{H}_\tau\to\mathscr{H}_t$是连续的. 因此，映射$\chi$是$\mathscr{H}_t$上

的共圈映射. 因为 $g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$满足式(7),故设 R_δ 是所有函数 $r:\mathbb{R}\to(0,+\infty)$的集合,满足

$$\lim_{t\to-\infty}\mathrm{e}^{\delta t}r^2(t)=0, \tag{13}$$

且 $D_{\delta,\mathscr{H}_t}$是对任意的 $r_D\in R_\delta$,使得 $D(t)\subset\overline{B}(0;r_D(t))$的所有 $\hat{D}=\{D(t),\ t\in\mathbb{R}\}$的集类,其中 $\overline{B}(0,r_D(t))$是以0为中心、$r_D(t)$为半径的闭球.

2.2 拉回吸收集的存在性

下面对方程(2)的解在空间 $\mathscr{H}_t$ 中做先验估计. 令 $\mathscr{K}(t)=\frac{1}{2}\varepsilon(t)\|u_t\|^2+\frac{1}{2}\|u\|^2_{V_2}$,定义能量函数

$$E(t)=\mathscr{K}(t)+\int_\Omega F(u)\mathrm{d}x. \tag{14}$$

由 $V_2\hookrightarrow L^{p+1}$和式(6)可得

$$\int_\Omega F(u)\mathrm{d}x\leqslant C\int_\Omega(|u|^2+|u|^{p+1})\mathrm{d}x\leqslant C(\|u\|^2+\|u\|^{p+1}_{p+1})\leqslant$$

$$C(\|u\|^2_{V_2}+\|u\|^{p+1}_{V_2})\leqslant Q(\|u\|^2_{V_2}), \tag{15}$$

其中 $Q(\cdot)$是递增函数. 结合式(8)和式(15)可知存在正常数 C_1,使得

$$C_1\mathscr{K}(t)-C\leqslant E(t)\leqslant Q(\mathscr{K}(t)). \tag{16}$$

将方程(2)与 $u_t+\delta u$ 在 L^2 中做内积,可得

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)+N(\|u\|^2_{V_1})\|u_t\|^2_{V_\theta}+\delta[\|u\|^2_{V_2}+\langle f(u),u\rangle]-\frac{\varepsilon'(t)}{2}\|u_t\|^2-\delta\varepsilon'(t)\langle u_t,u\rangle-$$

$$\delta\varepsilon(t)\|u_t\|^2-\delta N'(A^{1/4}u,A^{1/4}u_t)\|u\|^2_{V_\theta}=\langle g(t),u_t+\delta u\rangle, \tag{17}$$

其中

$$E_1(t)=E(t)+\delta\varepsilon(t)\langle u_t,u\rangle+\frac{\delta}{2}N(\|u\|^2_{V_1})\|u\|^2_{V_\theta}.$$

因为

$$\langle g(t),u_t+\delta u\rangle\leqslant\frac{a\delta^2}{4\lambda_1}\|u\|^2_{V_2}+\frac{a}{4}\|u_t\|^2+\frac{a\delta}{2}\langle u,u_t\rangle+\frac{1}{a}\|g(t)\|^2, \tag{18}$$

所以有

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)+\delta E_1(t)+G(t)\leqslant\frac{1}{a}\|g(t)\|^2, \tag{19}$$

其中

$$G(t)=N(\|u\|^2_{V_1})\|u_t\|^2_{V_\theta}+\left(-\frac{\varepsilon'(t)}{2}-\frac{3\delta\varepsilon(t)}{2}-\frac{\alpha}{4}\right)\|u_t\|^2+\delta\langle f(u),u\rangle-\delta\int_\Omega F(u)\mathrm{d}x+$$

$$\left(\frac{\delta}{2}-\frac{\alpha\delta^2}{4\lambda_1}\right)\|u\|^2_{V_2}-\delta N'(\|u\|^2_{V_1})(A^{1/4}u,A^{1/4}u_t)\|u\|^2_{V_\theta}-$$

$$\frac{\delta}{2}N(\|u\|^2_{V_1})\|u\|^2_{V_\theta}-\left(\delta\varepsilon'(t)+\frac{\alpha\delta}{2}+\delta^2\varepsilon(t)\right)\langle u_t,u\rangle. \tag{20}$$

由式(8),(9)可得

$$\delta\left[\langle f(u),u\rangle-\int_\Omega F(u)\mathrm{d}x\right]\geqslant-\frac{\delta\eta}{2\lambda_1}\|u\|^2_{V_2}-\delta C. \tag{21}$$

由 Young 不等式和 Hölder 不等式可得

$$|\delta N'(\|u\|^2_{V_1})\langle A^{1/4}u,A^{1/4}u_t\rangle|\leqslant C\delta|\langle A^{1/2}u,u_t\rangle|\leqslant\frac{\delta\eta}{2}\|u\|^2_{V_2}+\frac{\delta}{2\eta}\|u_t\|^2, \tag{22}$$

$$|\varepsilon'(t)\langle u_t,u\rangle|\leqslant L\|u_t\|\,\|u\|\leqslant\frac{L^2}{4\eta\lambda_1}\|u_t\|^2+\eta\|u\|^2\leqslant\frac{L^2}{4\eta\lambda_1}\|u_t\|^2+\frac{\eta}{\lambda_1}\|u\|^2_{V_2}, \tag{23}$$

$$|\varepsilon(t)\langle u_t,u\rangle|\leqslant\frac{L}{4}\|u_t\|^2+\frac{L\delta^2}{\lambda_1}\|u\|^2_{V_2}. \tag{24}$$

将式(21)~(24)代入式(20)可得

$$G(t)\geqslant N(\|u\|^2_{V_1})\|u_t\|^2_{V_\theta}+\left(\frac{\delta}{2}-\frac{\alpha\delta^2}{4\lambda_1}-\frac{\alpha\delta}{4}-\frac{L\delta^4}{\lambda_1}-\frac{\eta\delta}{2\lambda_1}-\frac{\delta\eta}{2}-\frac{\eta\delta}{\lambda_1}\right)\|u\|^2_{V_2}-\delta C+$$

$$\left(-\frac{\varepsilon'(t)}{2}-\frac{\alpha}{2}-\frac{\delta L^2}{4\eta\lambda_1}-\frac{\alpha\delta}{4}-\frac{\delta}{2\eta}-\frac{3\delta L}{2}\right)\|u_t\|^2. \tag{25}$$

取 δ,η 足够小，使得

$$\frac{\delta}{2}-\frac{\alpha\delta^2}{2\lambda_1}-\frac{\alpha\delta}{2}-\frac{L\delta^4}{\lambda_1}-\frac{\eta\delta}{2\lambda_1}-\frac{C(\delta)}{2}-\frac{\eta\delta}{\lambda_1}\geqslant 0,$$

$$-\frac{\varepsilon'(t)}{2}-\frac{\alpha}{2}-\frac{\delta L^2}{4\eta\lambda_1}-\frac{\alpha\delta}{4}-\frac{\delta}{2\eta}-\frac{3\delta L}{2}\geqslant 0,$$

从而有

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)+\delta E_1(t)\leqslant\delta C+\frac{2}{a}\|g(t)\|^2. \tag{26}$$

对式(26)两边同乘 $\mathrm{e}^{\delta t}$，并在$[t-\tau,t]$上积分可得

$$E_1(t)\leqslant\mathrm{e}^{-\delta\tau}E_1(t-\tau)+C(1-\mathrm{e}^{-\delta\tau})+\frac{2}{\alpha}\mathrm{e}^{-\delta t}\int_{t-\tau}^{t}\mathrm{e}^{-\delta s}\|g(s)\|^2\mathrm{d}s.$$

对任意的 $y_0\in D_{t-\tau}$，$t\in\mathbb{R}$，$\tau>0$，有

$$\sup_{y_0\in D_{t-\tau}}E_1(t-\tau)=\sup_{y_0\in D_{t-\tau}}\left\{\frac{1}{2}\|v_0\|_{V_2}^2+\frac{1}{2}\varepsilon(t-\tau)\|u_1\|^2+\int_\Omega F(u_0)\mathrm{d}x+\right.$$

$$\left.\delta\varepsilon(t-\tau)\langle u_0,v_0\rangle+\frac{\delta}{2}N(\|u_0\|_{V_1}^2)\|u_0\|_{V_\theta}^2\right\}<\infty.$$

从而对任意的 $y_0\in D_{t-\tau}$，$t\in\mathbb{R}$，$\tau>0$，有

$$\|\chi(\tau,t-\tau,y_0)\|_{\mathscr{H}_t}^2\leqslant 2C\mathrm{e}^{-\delta t}E_1(t-\tau)+2C(1-\mathrm{e}^{-\delta\tau})+\frac{2}{\alpha}\mathrm{e}^{-\delta t}\int_{t-\tau}^{t}\mathrm{e}^{-\delta s}\|g(s)\|^2\mathrm{d}s,$$

所以 $\|\chi(\tau,t-\tau,y_0)\|_{\mathscr{H}_t}^2$ 有界. 假设

$$(R_\delta(t))^2=2C(1-\mathrm{e}^{-\delta\tau})+\frac{2}{\alpha}\mathrm{e}^{-\delta t}\int_{t-\tau}^{t}\mathrm{e}^{-\delta s}\|g(s)\|^2\mathrm{d}s,$$

考虑 $\mathscr{H}_t$ 中的闭球 $\hat{B}_{\delta,\mathscr{H}_t}$，定义 $\hat{B}_{\delta,\mathscr{H}_t}=\{y\in\mathscr{H}_t:\|y\|_{\mathscr{H}_t}^2\leqslant(R_\delta(t))^2\}$，结合式(13)可知，$\hat{B}_{\delta,\mathscr{H}_t}$ 是共圈 χ 的拉回 $D_{\delta,\mathscr{H}_t}$ 吸收集.

定理 4　假设条件(H_1)～(H_4)成立，则方程(2)产生的非自治动力系统(θ,χ)在 $\mathscr{H}_t$ 中存在拉回 $D_{\delta,\mathscr{H}_t}$ 吸收集.

证明：由先验估计可直接得出方程(2)产生的非自治动力系统(θ,χ)在 $\mathscr{H}_t$ 中存在拉回 $D_{\delta,\mathscr{H}_t}$ 吸收集.

3　拉回吸引子的存在性

引理 1　假设条件(H_1)～(H_4)成立，则定义在 $X\times X$ 上的函数 $\Phi(\cdot,\cdot)$是收缩函数.

证明：对任意的 $t\in\mathbb{R}$，令 $y_i=(u_i(t),u_{it}(t))(i=1,2)$为问题(2)关于初值 $y_0^i=(u_0^i,v_0^i)\in D_{t-\tau}\times D_{t-\tau}$的解，其中 $i>0$. 记 $w(t)=u_1(t)-u_2(t)$，且 $w(t)$满足：

$$\begin{cases}\varepsilon(t)w_{tt}+Aw+Y_{12}A^{\theta/2}w_t+\widetilde{Y}_{12}A^{\theta/2}(u_{1t}+u_{1t})+f(u_1)-f(u_2)=\\ \qquad g_1(x,t)-g_2(x,t),\quad x\in\Omega,\quad t\geqslant\tau,\\ w(x,\tau)=u_0^1-u_0^2,\quad w_t(x,\tau)=v_0^1-v_0^2,\quad x\in\Omega,\end{cases} \tag{27}$$

其中

$$Y_{12}=\frac{1}{2}(N(\|u_1\|_{V_1}^2)+N(\|u_2\|_{V_1}^2)),$$

$$\widetilde{Y}_{12}=\frac{1}{2}\int_0^1 N'(\lambda\|u_1\|_{V_1}^2+(1-\lambda)\|u_2\|_{V_1}^2)\mathrm{d}\lambda\langle A^{1/4}(u_1+u_2),A^{1/4}w\rangle.$$

记

$$\mathscr{G}_w(t)=\frac{1}{2}\int_\Omega(\varepsilon(t)|w_t|^2+|A^{1/2}w|^2)\mathrm{d}x.$$

用 $\mathrm{e}^{\beta t}w_t(t)$与式(27)在 $L^2(\Omega)$中做内积，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\beta t}\mathscr{G}_w(t)-\frac{\mathrm{e}^{\beta t}\varepsilon'(t)}{2}\|w_t\|^2+\int_\Omega \mathrm{e}^{\beta t}Y_{12}|A^{\theta/4}w_t|^2\mathrm{d}x+$$

$$\int_\Omega \mathrm{e}^{\beta t}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t\mathrm{d}x+\int_\Omega \mathrm{e}^{\beta t}(f(u_1)-f(u_2))w_t\mathrm{d}x=$$

$$\beta\mathrm{e}^{\beta t}\mathscr{G}_w(t)+\int_\Omega \mathrm{e}^{\beta t}(g_1-g_2)w_t\mathrm{d}x. \tag{28}$$

将式(28)在$[s,t]$上积分，并由$\varepsilon'(\cdot)<0$可得

$$\mathrm{e}^{\beta t}\mathscr{G}_w(t)-\mathrm{e}^{\beta t}\mathscr{G}_w(s)+\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}Y_{12}|A^{\theta/4}w_t(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta+\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta+$$

$$\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\leqslant$$

$$\beta\int_s^t \mathrm{e}^{\beta\zeta}\mathscr{G}_w(\zeta)\mathrm{d}\zeta+\int_s^t\int_\Omega \mathrm{e}^{\beta t}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta. \tag{29}$$

将式(29)关于s在$[t-\tau,t]$上积分，有

$$\tau\mathrm{e}^{\beta t}\mathscr{G}_w(t)-\int_{t-\tau}^t \mathrm{e}^{\beta s}\mathscr{G}_w(s)\mathrm{d}s+\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+$$

$$\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}Y_{12}|A^{\theta/4}w_t(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+$$

$$\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s\leqslant$$

$$\beta\int_{t-\tau}^t\int_s^t \mathrm{e}^{\beta\zeta}\mathscr{G}_w(\zeta)\mathrm{d}\zeta\mathrm{d}s+\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta t}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s. \tag{30}$$

用$\mathrm{e}^{\beta t}w(t)$与式(27)在$L^2(\Omega)$中做内积，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\beta t}\varepsilon(t)\langle w,w_t\rangle+\int_\Omega \mathrm{e}^{\beta t}|A^{1/2}w|^2\mathrm{d}x-\int_\Omega \mathrm{e}^{\beta t}\varepsilon(t)|w_t|^2\mathrm{d}x-\mathrm{e}^{\beta t}(\varepsilon'(t)+\beta\varepsilon(t))\langle w,w_t\rangle=$$

$$-\int_\Omega \mathrm{e}^{\beta t}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w\mathrm{d}x-\int_\Omega \mathrm{e}^{\beta t}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w\mathrm{d}x+$$

$$\int_\Omega \mathrm{e}^{\beta t}(g_1-g_2)w\mathrm{d}x-\int_\Omega \mathrm{e}^{\beta t}(f(u_1)-f(u_2))w\mathrm{d}x. \tag{31}$$

对式(31)在$[s,t]$上积分，有

$$\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}|A^{1/2}w(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta+\int_\Omega \mathrm{e}^{\beta t}\varepsilon(t)w_t(t)w(t)\mathrm{d}x=$$

$$\int_\Omega \mathrm{e}^{\beta s}\varepsilon(s)w_t(s)w(s)\mathrm{d}x+\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta+\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(g_1-g_2)w(\zeta)\mathrm{d}x\mathrm{d}\zeta+$$

$$\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta-$$

$$\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta. \tag{32}$$

对式(32)关于s在$[t-\tau,t]$上积分，并在其左右两边同时乘以β，有

$$\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}|A^{1/2}w(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=$$

$$\beta\int_{t-\tau}^t\int_\Omega \mathrm{e}^{\beta s}\varepsilon(s)w_t(s)w(s)\mathrm{d}x\mathrm{d}s+2\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+$$

$$\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\varepsilon(\zeta)(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\beta\tau\int_\Omega \mathrm{e}^{\beta t}\varepsilon(t)w_t(t)w(t)\mathrm{d}x-$$

$$\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(g_1-g_2)w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-$$

$$\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-$$

$$\beta\int_{t-\tau}^t\int_s^t\int_\Omega \mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s. \tag{33}$$

将式(33)代入式(30)可得

$$\begin{aligned}\tau e^{\beta t}\mathscr{G}_w(t)-\int_{t-\tau}^{t}e^{\beta s}\mathscr{G}_w(s)ds\leqslant{}&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)w_t(\zeta)w(\zeta)dxd\zeta+\beta\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2dxd\zeta ds+\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)dxd\zeta ds-\frac{\beta}{2}\tau\int_{\Omega}e^{\beta t}\varepsilon(t)w_t(t)w(t)dx-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)dxd\zeta ds+\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w(\zeta)dxd\zeta ds-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)dxd\zeta ds-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}|A^{\theta/4}w_t(\zeta)|^2dxd\zeta ds-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)dxd\zeta ds-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w_t(\zeta)dxd\zeta ds-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)dxd\zeta ds-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)dxd\zeta ds.\end{aligned}\tag{34}$$

将式(31)在$[t-\tau,t]$上积分可得

$$\begin{aligned}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}|A^{1/2}w(\zeta)|^2dxd\zeta={}&\int_{\Omega}e^{\beta(t-\tau)}\varepsilon(t-\tau)w_t(t-\tau)w(t-\tau)dx+\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2dxd\zeta-\\&\int_{\Omega}e^{\beta t}\varepsilon(t)w_t(t)w(t)dx+\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)dxd\zeta-\\&\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)dxd\zeta-\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{1t})\cdot A^{\theta/4}w(\zeta)dxd\zeta-\\&\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2)w(\zeta)dxd\zeta+\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w(\zeta)dxd\zeta.\end{aligned}\tag{35}$$

将式(35)代入式(34)可得

$$\begin{aligned}\tau e^{\beta t}\mathscr{G}_w(t)+\int_{t-\tau}^{t}e^{\beta s}\mathscr{G}_w(s)ds\leqslant{}&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)w_t(\zeta)w(\zeta)dxd\zeta+\beta\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2dxd\zeta ds+\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)dxds-\left(1+\frac{\beta\tau}{2}\right)\int_{\Omega}e^{\beta t}\varepsilon(t)w_t(t)w(t)dx-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)dxd\zeta ds+\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w(\zeta)dxd\zeta-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)dxd\zeta ds-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}|A^{\theta/4}w_t(\zeta)|^2dxd\zeta ds-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)dxd\zeta ds+\int_{\Omega}e^{\beta(t-\tau)}\varepsilon((t-\tau))w_t(t-\tau)w(t-\tau)dx+\\&2\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\varepsilon(\zeta)|w_t(\zeta)|^2dxd\zeta-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)dxd\zeta ds-\\&\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)dxd\zeta-\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)dxd\zeta-\\&\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)dxd\zeta+\int_{t-\tau}^{t}\int_{\Omega}e^{\beta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)dxd\zeta-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)dxd\zeta ds-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w_t(\zeta)dxd\zeta ds.\end{aligned}\tag{36}$$

将式(28)在$[t-\tau,t]$上积分，并结合$\varepsilon'(\cdot)<0$可得

$$\begin{aligned}-\int_{t-\tau}^{t}e^{\beta\zeta}\mathscr{G}_w(\zeta)d\zeta\leqslant{}&\frac{1}{\beta}e^{\beta(t-\tau)}\mathscr{G}_w(t-\tau)-\frac{1}{\beta}e^{\beta t}\mathscr{G}_w t-\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}Y_{12}|A^{\theta/4}w_t(\zeta)|^2dxd\zeta+\\&\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}(g_1-g_2)w_t(\zeta)dxd\zeta-\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}e^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)dxd\zeta-\end{aligned}$$

$$\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta-$$

$$\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta. \tag{37}$$

将式(37)代入式(36),有

$$\begin{aligned}\tau\mathrm{e}^{\beta t}\mathscr{G}_w(t)\leqslant&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta+\beta\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)\,|\,w_t(\zeta)\,|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\left(1+\frac{\beta\tau}{2}\right)\int_{\Omega}\mathrm{e}^{\beta t}\varepsilon(t)w_t(t)w(t)\mathrm{d}x-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\\&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\,|\,A^{\theta/4}w_t(\zeta)\,|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\,|\,A^{\theta/4}w_t(\zeta)\,|^2\mathrm{d}x\mathrm{d}\zeta+\\&2\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)\,|\,w_t(\zeta)\,|^2\mathrm{d}x\mathrm{d}\zeta-\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\\&\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_t\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\\&\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta+\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\\&\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta-\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta+\\&\frac{1}{\beta}\mathrm{e}^{\beta(t-\tau)}\mathscr{G}_w(t-\tau)+\int_{\Omega}\mathrm{e}^{\beta(t-\tau)}\varepsilon((t-\tau))w_t(t-\tau)w(t-\tau)\mathrm{d}x+\\&\frac{1}{\beta}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta+\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\\&\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s.\end{aligned} \tag{38}$$

下面估计式(38)中的一些项:

$$\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta\leqslant\frac{\beta L}{4}\left(\mathrm{e}^{\beta t}\parallel w(t)\parallel^2+\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2x\mathrm{d}\zeta\right), \tag{39}$$

$$\begin{aligned}&\frac{\beta}{2}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(\varepsilon'(\zeta)+\beta\varepsilon(\zeta))w_t(\zeta)w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s\leqslant\\&\frac{\beta L}{4}\int_{t-\tau}^{t}(\mathrm{e}^{\beta t}\parallel w(t)\parallel^2-\mathrm{e}^{\beta s}\parallel w(s)\parallel^2-\beta\tau\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2)\mathrm{d}\zeta+\frac{\beta^2}{4}\Big(\tau\varepsilon(t)\mathrm{e}^{\beta t}\parallel w(t)\parallel^2-\\&\int_{t-\tau}^{t}\varepsilon(\zeta)\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta+\tau L\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta-\beta\tau\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta\Big)\leqslant\\&\frac{\beta^2\tau L}{4}\mathrm{e}^{\beta t}\parallel w(t)\parallel^2-\frac{\beta L}{4}(1+\beta\tau)\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta+\frac{\beta^2\tau L}{4}\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta=\\&\left(\frac{\beta^2\tau L}{4}-\frac{\beta L}{4}(1+\beta\tau)\right)\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta+\frac{\beta^2\tau L}{4}\mathrm{e}^{\beta t}\parallel w(t)\parallel^2.\end{aligned} \tag{40}$$

结合式(38)~(40),令

$$\begin{aligned}\Phi_{t,\tau}((u_0^1,v_0^1),(u_0^2,v_0^2))=&-\frac{\beta L\mathrm{e}^{-\beta t}}{2}\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta+\\&\frac{\mathrm{e}^{-\beta t}}{\tau}\left(\frac{\beta^2\tau L}{4}-\frac{\beta L}{4}(1+\beta\tau)\right)\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta+\\&\frac{\beta\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)\,|\,w_t(\zeta)\,|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+\frac{\beta L\mathrm{e}^{-\beta t}}{4\tau}\left(\mathrm{e}^{\beta t}\parallel w(t)\parallel^2+\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\parallel w(\zeta)\parallel^2\mathrm{d}\zeta\right)-\end{aligned}$$

$$\frac{(1+\beta\tau)\mathrm{e}^{-\beta t}}{2\tau}\int_{\Omega}\mathrm{e}^{\beta t}\varepsilon(t)w_t(t)\mathrm{d}x-\frac{\mathrm{e}^{-\beta t}}{\beta\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta-$$
$$\frac{\beta\mathrm{e}^{-\beta t}}{2\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_tA^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\frac{\mathrm{e}^{-\beta t}}{\beta\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\left|A^{\theta/4}w_t(\zeta)\right|^2\mathrm{d}x\mathrm{d}\zeta-$$
$$\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-\frac{\beta L\mathrm{e}^{-\beta t}}{2}\int_{t-\tau}^{t}\mathrm{e}^{\beta\zeta}\|w(\zeta)\|^2\mathrm{d}\zeta-$$
$$\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s+\frac{\beta^2L}{4}\|w(t)\|^2-$$
$$\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}w_tA^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta-\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta+$$
$$\frac{\mathrm{e}^{-\beta t}}{\beta\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta-\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\left|A^{\theta/4}w_t(\zeta)\right|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-$$
$$\frac{\mathrm{e}^{-\beta t}}{\beta\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta+\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w(\zeta)\mathrm{d}x\mathrm{d}\zeta-$$
$$\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta+\frac{2\mathrm{e}^{-\beta t}}{\tau}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)\left|w_t(\zeta)\right|^2\mathrm{d}x\mathrm{d}\zeta-$$
$$\frac{\beta\mathrm{e}^{-\beta t}}{2\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_1)-f(u_2))w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-$$
$$\frac{\beta\mathrm{e}^{-\beta t}}{2\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{1t}+u_{2t})\cdot A^{\theta/4}w(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s-$$
$$\frac{\mathrm{e}^{-\beta t}}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(g_1-g_2)w_t(\zeta)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s,\tag{41}$$

则有

$$\mathscr{G}_w(t)\leqslant C\frac{\mathrm{e}^{-\beta t}}{\tau}\widetilde{R}^2_{t-\tau}+\Phi_{t,\tau}((u_0^1,v_0^1),(u_0^2,v_0^2)),\tag{42}$$

其中

$$\widetilde{R}^2_{t-\tau}=\frac{1}{\beta}\mathrm{e}^{\beta(t-\tau)}\mathscr{G}_w(t-\tau)+\int_{\Omega}\mathrm{e}^{\beta(t-\tau)}\varepsilon((t-\tau))w_t(t-\tau)w(t-\tau)\mathrm{d}x.$$

令(u_n,u_{nt})是对应于初值$(u_0^i,u_{10}^i)\in D_{t-\tau_0}\times D_{t-\tau_0}$的解. 已知$\|u_n\|^2_{V_2}$有界，由条件$(\mathrm{H}_1)$可知，对任意固定的$t-\tau_0$，存在$\zeta\in[t-\tau_0,t]$，使得$\varepsilon(\zeta)$有界，所以$\varepsilon(\zeta)\|u_{nt}\|^2$有界，因此$\|u_n\|^2_{V_2}+\varepsilon(\zeta)\|u_{nt}\|^2$有界. 根据 Alaoglu 定理可知，$u_n\to u$ 在$L^{\infty}([t-\tau_0,t];V_2)$中弱$*$收敛，$\partial_tu_n\to\partial_tu$ 在$L^2([t-\tau_0,t];V_\theta)$中弱收敛，$\partial_tu_n\to\partial_tu$ 在$L^{\infty}((t-\tau_0,t];L^2)$中弱$*$收敛，在$L^2([t-\tau_0,t];V_2)$中$u_n\to u$，在$L^2([t-\tau_0,t];L^2)$中$\partial_tu_n\to\partial_tu$，在$L^2([t-\tau_0,t];V_{2\theta})$中$u_n\to u$，其中用到了$\theta<2\theta<2$. 此外，因为$2\leqslant p<p_\theta=5+4\theta$，所以有嵌入$V_2\hookrightarrow\hookrightarrow L^2$，$V_2\hookrightarrow L^{p+1}$. 下面依次处理式(41)中的每一项：

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\left|u_{nt}-u_{mt}\right|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{nt}+u_{mt})A^{\theta/4}(u_{nt}-u_{mt})\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{nt}+u_{mt})A^{\theta/4}(u_{nt}-u_{mt})\mathrm{d}x\mathrm{d}\zeta=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{s}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\left|A^{\theta/4}(u_{nt}-u_{mt})\right|^2\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\varepsilon(\zeta)\left|(u_{nt}-u_{mt})\right|^2\mathrm{d}x\mathrm{d}\zeta=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}\left|A^{\theta/4}u_{nt}-u_{mt}\right|^2\mathrm{d}x\mathrm{d}\zeta=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{\Omega}\mathrm{e}^{\beta t}\varepsilon(t)(u_{nt}-u_{mt})\mathrm{d}x=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|u_n-u_m\|^2\mathrm{d}\zeta\leqslant\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|u_n-u_m\|_{V_2}^2\mathrm{d}x\mathrm{d}\zeta=0.$$

由 Hölder 不等式和 Sobolev 嵌入可得

$$\lim_{n\to\infty}\lim_{m\to\infty}\left|\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_n)-f(u_m))\cdot(u_{n_t}-u_{m_t})\mathrm{d}x\mathrm{d}\zeta\right|\leqslant$$

$$C\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}(1+\|u_n\|_{\frac{p-1}{p+1}}+\|u_m\|_{\frac{p-1}{p+1}})\|u_n-u_m\|_{\frac{p-1}{p+1}}\|u_{n_t}-u_{m_t}\|_{\frac{p-1}{p+1}}\mathrm{d}\zeta\leqslant$$

$$C\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}(1+\|u_n\|_{V_2}^{\frac{p-1}{V_2}}+\|u_m\|_{V_2}^{\frac{p-1}{V_2}})\|u_n-u_m\|_{V_2}^{\frac{p-1}{V_2}}\|u_{n_t}-u_{m_t}\|_{V_2}^{\frac{p-1}{V_2}}\mathrm{d}\zeta=0.\quad(43)$$

对于固定的 t，$\left|\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_n(\xi))-f(u_m(\zeta)))(u_{n_t}(\zeta)-u_{m_t}(\zeta))\mathrm{d}x\mathrm{d}\zeta\right|$ 有界，由 Lebesgue 控制收敛定理可得

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_n(\zeta))-f(u_m(\zeta)))(u_{n_t}(\zeta)-u_{m_t}(\zeta))\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=$$

$$\int_{t-\tau_0}^{t}\left(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_{\Omega}\mathrm{e}^{\gamma\zeta}(f(u_n(\zeta))-f(u_m(\zeta)))(u_{n_t}(\zeta)-u_{m_t}(\zeta))\mathrm{d}x\mathrm{d}\zeta\right)\mathrm{d}s=0.$$

类似地，有

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}(f(u_n)-f(u_m))(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}(u_{nt}-u_{mt})A^{\theta/4}(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\leqslant$$

$$C\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|(u_{nt}-u_{mt})\|_{V_\theta}\|(u_n-u_m)\|_{V_\theta}\mathrm{d}\zeta=0.$$

对于固定的 t，$\left|\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}(u_{nt}-u_{mt})A^{\theta/4}(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\right|$ 有界，由 Lebesgue 控制收敛定理可得

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}Y_{12}A^{\theta/4}(u_{nt}-u_{mt})A^{\theta/4}(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{nt}+u_{mt})A^{\theta/4}(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\leqslant$$

$$C\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|u_n-u_m\|\int_{t-\tau_0}^{t}|A^{\theta/4}(u_{nt}+u_{mt})||A^{\theta/4}(u_n-u_m)|\mathrm{d}x\mathrm{d}\zeta\leqslant$$

$$C\lim_{n\to\infty}\lim_{m\to\infty}\Big(\sup_{\zeta\in[t-\tau_0,t]}\mathrm{e}^{\beta\zeta}\|u_n-u_m\|\Big)\Big(\int_{t-\tau_0}^{t}(\|u_{nt}\|_{V_\theta}^2+\|u_{nt}\|_{V_\theta}^2)\mathrm{d}\zeta\Big)^{1/2}\times$$

$$\Big(\int_{t-\tau_0}^{t}\|u_n-u_m\|_{V_\theta}^2\mathrm{d}\zeta\Big)^{1/2}=0.$$

进而有

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}\widetilde{Y}_{12}A^{\theta/4}(u_{nt}+u_{mt})A^{\theta/4}(u_n-u_m)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0.$$

因为

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}|g_n-g_m||u_{nt}-u_{mt}|\mathrm{d}x\mathrm{d}\zeta=0,$$

再结合 Lebesgue 控制收敛定理可得

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_s^t\int_{\Omega}\mathrm{e}^{\beta\zeta}|g_n-g_m||u_{nt}-u_{mt}|\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0,$$

而

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\int_{\Omega}\mathrm{e}^{\beta\zeta}|g_n-g_m||u_n-u_m|\mathrm{d}x\mathrm{d}\zeta\leqslant\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|g_n-g_m\|\,\|u_n-u_m\|\mathrm{d}\zeta\leqslant$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t-\tau_0}^{t}\mathrm{e}^{\beta\zeta}\|g_n-g_m\|\,\|u_n-u_m\|_{V_2}\mathrm{d}\zeta=0,$$

因此可知 $\Phi_{t,\tau_0}((u_0^1,v_0^1),(u_0^2,v_0^2))$为 $D_{t-\tau_0}\times D_{t-\tau_0}$ 上的收缩函数.

定理 5　假设条件(H_1)～(H_4)成立，则由问题(2)的解生成的非自治动力系统(θ,χ)在 $\mathscr{H}_t$ 中存在时间依赖拉回吸引子.

证明：由引理 1 知，$\Phi_{t,\tau_0}(\cdot,\cdot)$是一个收缩函数，结合定理 2 和定理 3 可知，问题(2)的解生成的非自治动力系统(θ,χ)在 $\mathscr{H}_t$ 中存在时间依赖拉回吸引子.

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