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Research Progress of the Sampling Theorem Associated with the Fractional Fourier Transform

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Abstract: Sampling is a bridge between continuous-time and discrete-time signals, which is important to digital signal processing. The fractional Fourier transform (FrFT) that serves as a generalization of the FT can characterize signals in multiple fractional Fourier domains, and therefore can provide new perspectives for signal sampling and reconstruction. In this paper, we review recent developments of the sampling theorem associated with the FrFT, including signal reconstruction and fractional spectral analysis of uniform sampling, nonuniform samplings due to various factors, and sub-Nyquist sampling, where bandlimited signals in the fractional Fourier domain are mainly taken into consideration. Moreover, we provide several future research topics of the sampling theorem associated with the FrFT.

Keywords: fractional Fourier transform; nonuniform sampling; signal reconstruction; spectral analysis; uniform sampling

1 Introduction

The well-known Nyquist sampling theorem shows that a continuous-time signal can be uniquely determined by its uniform samples if it is bandlimited in the Fourier domain and the sampling rate is at least the Nyquist rate [1]. However, the signals that are non-bandlimited in the Fourier domain are commonly encountered and the Nyquist rate is usually too high to achieve in practice. Both cases necessitate a high sampling rate to enable the performance of signal reconstruction and spectral analysis, which can result in large amount of samples and high computational complexity in the digital signal processing. In order to reduce the computational complexity in practice, the sampling may not meet the requirement of the Nyquist rate, which therefore degrades the performance of signal processing.

To solve this problem, we can resort to generalized signal processing method to develop advanced sampling theorem. As a generalization of the classical Fourier transform (FT), the fractional Fourier transform (FrFT) has been widely applied to radar [2-6], communications [7, 8], optics [9, 10], and image signal processing [11, 12], etc. This is because unlike the FT which can only characterize signals in the frequency domain, the FrFT can characterize signals in multiple fractional Fourier domains including the time and the frequency domains, and therefore, can provide new perspectives for signal processing. Thus the FrFT has to be integrated in the study of sampling theorem.

At first, the research on the sampling theorem associated with the FrFT focused on the signal reconstruction and spectral analysis with respect to the uniform sampling [13-17]. Moreover, nonuniform sampling theorem associated with

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the FrFT has been widely studied. On the one hand, this is because nonuniform sampling commonly occurs in various applications such as radar [18], communications [19], optics [20], astronomy [21], and seismology [22], which may degrade the fidelity of signals and the signal-to-noise ratio of signal processing systems [23]. On the other hand, the proper use of nonuniform sampling can help to reduce the requirement of sampling rate and obtain desirable characteristics of signals [23]. What 's more, sub-Nyquist sampling associated with the FrFT has also been an important research topic, based on which the sampling rate and computational complexity can be greatly decreased.

In this paper, we review recent developments of the sampling theorem associated with the FrFT. The rest of the paper is organized as follows. In Section 2, the uniform sampling theorems associated with the FrFT are reviewed. In Section 3, the nonuniform sampling theorems associated with the FrFT are given. The sub-Nyquist sampling theorems associated with the FrFT are offered in Sections 4. Several future research topics are provided in Section 5 and Section 6 concludes the paper.

2 Uniform Sampling

2.1 Signal Reconstruction and Fractional Spectral Analysis for Uniform Sampling

Xia first studied the uniform sampling theorem of bandlimited signals in the fractional Fourier domain in 1996 [13]. In order to obtain the interpolation formula of a uniformly sampled signal that is bandlimited in the fractional Fourier domain, the original signal is first modulated to obtain a signal that is bandlimited in the Fourier domain, and then the Shannon sampling theorem is applied to the modulated signal. In the following, there have been many works in literature that obtain the same interpolation formula by means of representing the FrFT by a variation of the FT [14], introducing the concept of chirp-periodicity and FrFT series [15], or studying the generalized Parseval formula based on the FrFT series [24]. In 2003, the work of [16] has developed the reconstruction formula of a signal observed within limited duration based on the FrFT series [16]. In 2005, the reconstruction formula of the signal that is periodic in the time domain and bandlimited in the fractional Fourier domain has been given in [25]. In 2012, the sampling theorem in the shift-variant space associated with the FrFT has been studied in [26].

When it comes to the random signals, the sampling theorem is based on fractional correlation function (FrCF) and fractional power spectral density (FrPSD) [27-29]. In 2010, the work of [30] has studied the uniform sampling theorem of chirp-stationary random signals that are bandlimited in the fractional Fourier domain and obtained the reconstruction formula according to Xia 's result. Based on [30], the work of [31] has established the relationship between the fractional sampling theorem for bandlimited random signals and von Neumann Ergodic theorem, and the work of [32] has analyzed the aliasing and truncation error of signal reconstruction. For more related results, see [33] and [34].

The work of [15] has showed that the uniform sampling in the time domain leads to the chirp-periodicity of the spectrum in the fractional Fourier domain based on the FrFT series. The work of [17] has obtained the expression of the fractional spectrum of a uniformly sampled signal by means of representing the sampled signal by the multiplication of the original signal and the impulse train, and then using the fractional convolution theorem. It shows that the fractional spectrum of a uniformly sampled signal is the periodic replicas of that of the original signal with a period of $\omega_s \sin\alpha$, along with linear phase modulation [17]. And therefore, the spectrum of a bandlimited signal in the fractional Fourier domain can be perfectly reconstructed by an ideal low-pass filter in the fractional Fourier domain when the replicas do not overlap, which requires that the signal is bandlimited to u_α in the frac-

tional Fourier domain and the sampling rate is at least $2u_\alpha \csc\alpha$. The uniform sampling theorem proposed by [17] can greatly decrease the computational complexity in the parameter estimation of optical fringes with quadratic phase, which has been evident in interferometric measurements with reduction of the processing time from 179 s to 0.83 s when the measured curvature radius is 0.86 m [10]. For more related results for the spectral analysis of sampling in the linear canonical domain, see [35, 36].

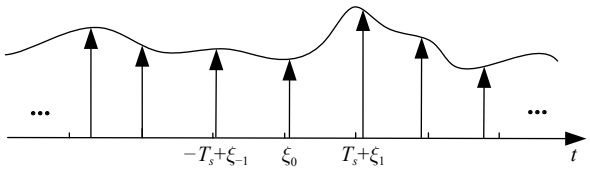
2.2 Sampling Rate Conversion in the Fractional Fourier Domain

Upsampling and downsampling of discrete-time signals are usually discussed in multirate signal processing for sampling rate conversion [37]. In 2006, the interpolation and decimation of a discrete-time signal bandlimited in the fractional Fourier domain have been studied in [17, 38], wherein the expressions of the fractional spectra of a discrete-time signal with interpolation or decimation by an integer factor have been derived. By designing filters in the fractional Fourier domain, the sampling rate conversion by a rational factor has also been realized, which paves the way for the multirate signal processing in the fractional Fourier domain [17]. For more details associated with the Linear canonical transformation, see [39]. In 2008, the fractional Fourier analyses of a signal with cyclic interpolation and decimation have been studied in [40], based on which the transform order division multiplexing has been developed to improve the performance of traditional multiplexing in the fractional Fourier domain that only transmits the signals in a single direction in the time-frequency plane [41].

3 Nonuniform Sampling

3.1 Signal Reconstruction and Fractional Spectral Analysis for Nonuniform Sampling Due to Clock Jitter/Random Perturbation

In high speed data acquisition system, clock jitter/random perturbation is usually caused by hot noise, shot noise, and flick noise, which will result in nonuniform sampling (see Fig. 1).



T_s is the sampling period and ξ_n, $n \in \mathbb{Z}$ is the nth clock jitter/random perturbation

Fig. 1 Nonuniform sampling due to clock jitter/random perturbation

Aiming at the deterministic signals that are bandlimited in the fractional Fourier domain, the work of [42] has constructed the frames of sampling and reconstruction in the functional space associated with the FrFT, based on which the irregular sampling theorem in the presence of random perturbation has been derived. Moreover, the work of [43] has extended the classical window-function-based method [44, 45] to the fractional Fourier domain, based on which the fractional spectral biases caused by clock jitter can be compensated using a window function. The drawback of this method is that the resulted window is a function of both the time and the fractional frequency. To tackle this drawback, the work of [23] has proposed a fractional spectral compensation method with both high accuracy and low computational complexity by means of developing optimal filter that minimizes the mean square error between the original and the compensated fractional spectra.

As for the random signals that are bandlimited in the fractional Fourier domain, the work of [46] has studied the signal reconstruction in the presence of random perturbation based on the Lagrange interpolation and the randomized sinc interpolations proposed in [47]. Although the Lagrange interpolation provides excellent reconstructions, it will result in high computational complexity because the forms of interpolators depend on the nonuniform sampling instants [23]. Furthermore, although the randomized-sinc-interpolation-based method is simple, it lacks accur-

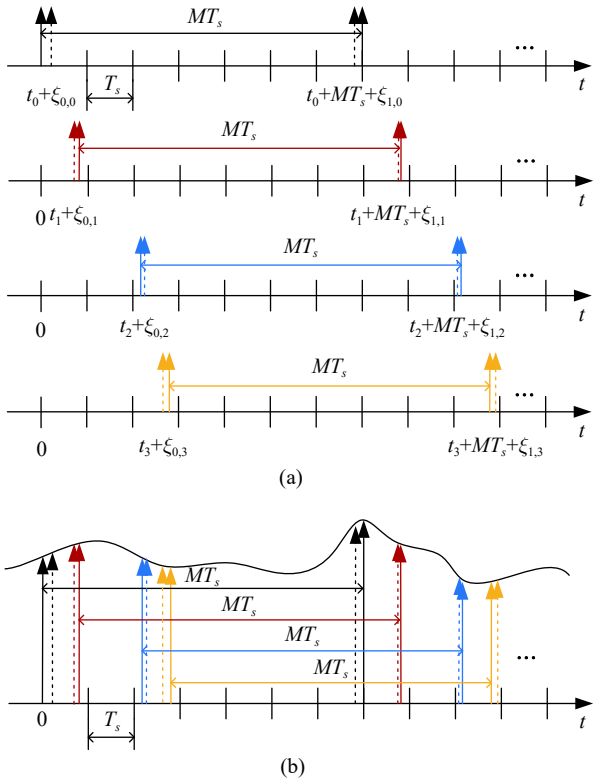
acy because the obtained reconstructed signals are the approximations of the original signal, which degrades the performance of the estimation of the FrPSD and FrCF. Moreover, a FrPSD estimation method has been proposed based on the analysis of the second-order random statistic characters, where the effect of random perturbation on the FrPSD has been equivalent to a prefilter in the fractional Fourier domain [46]. However, this method [46] requires the knowledge of the nonuniform sampling instants, which limits its applications. To solve this problem, the work of [48] has proposed estimations of FrPSD and FrCF that do not rely on the sampling instants, which cancel the biases of FrPSD and FrCF approximations by means of fractional inverse filtering and fractional interpolation, respectively. For more related works associated with the linear canonical transformation, see [49].

3.2 Signal Reconstruction and Fractional Spectral Analysis for Periodic Nonuniform Sampling Due to Timing Offset

Periodic nonuniform sampling due to timing offset was established in [50]. In general, M parallel channels can be used to sample a signal with a period of MT_s, respectively, and then all the output samples are multiplexed to obtain a sampling with an average period of T_s [47]. Timing offset will result in asynchronism of multiplexing analog-to-digital convertors (ADCs) and therefore lead to periodic nonuniform sampling (see Fig. 2, solid arrows).

For the signals that are bandlimited in the fractional Fourier domain, the work of [51, 52] have derived the fractional spectral expressions of a periodically nonuniformly sampled signal by the sum of the fractional spectra of the subsequences or by the computation of the discrete-time FrFT of the nonuniform sampled signal [23]. Based on the latter expression, the corresponding fractional spectral reconstruction method has been proposed by means of solving a system of equations [51, 52]. Furthermore, the work of [53] has extend the fractional spectral reconstruction method in [51, 52] to 2-D signals and applied it to synthetic aperture radar imaging. Based on Radon nonuniform FrFT, the work of [5] has proposed a coherent integration method for random pulse repetition interval radar, which can be utilized to tackle the problems of range cell migration and spectral spread.

When it comes to the interleaving/multiplexing ADCs in the presence of both the timing offset and the clock jitter (see Fig. 2, dashed arrows), the work of [23] has proposed a fractional spectral estimation with high accuracy by means of first decoupling the effect of them, and then compensating the biases due to timing offset and clock jitter by solving a fractional system of linear equations and developing optimal filter, respectively. In aforementioned works, the timing



M is the number of sampling channels, t_m and $\xi_{n,m}$, $m = 0, \cdots, M-1$, $n \in \mathbb{Z}$ are the first sampling instant only in the presence of the timing offset and the nth clock jitter/random perturbation, respectively, all associated with the mth channel

Fig. 2 Periodic nonuniform sampling due to timing offset (solid arrows, $t_m + nMT_s$) and nonuniform sampling in the presence of both the timing offset and the clock jitter (dashed arrows, $t_m + nMT_s + \xi_{n,m}$): (a) Sampling instants of each channel; (b) nonuniform samples

offset is assumed to be deterministic. As for random timing offset, the work of [43] has studied the expression and reconstruction of the statistical mean of the fractional spectrum.

3.3 Signal Reconstruction for Other Nonuniform Samplings

When the perturbations of sampling instants are less than $\frac{1}{4}$ sampling period, then the original signal can be reconstructed by Lagrange interpolation [54, 55]. As for the signals that are bandlimited in the fractional Fourier domain, the Lagrange interpolation associated with the FrFT has been studied in [52]. Moreover, the work of [56, 57] has studied the Lagrange interpolation in the reproducing kernel Hilbert space associated with the linear canonical transform. And the work of [46] has studied the Lagrange interpolation of random signals that are bandlimited in the fractional Fourier domain. The main advantage of Lagrange interpolation is that it is capable of providing excellent reconstruction, while the disadvantage is that the interpolators rely on the nonuniform instants, and therefore will result in high computational complexity [23]. Note also that such nonuniform scheme can reduce to certain type of aforementioned nonuniform schemes when the condition that the perturbations of sampling instants are less than $\frac{1}{4}$ sampling period is satisfied. For example, the work of [46] has utilized the Lagrange interpolation associated with the FrFT to reconstruct a signal from nonuniform sampling due to clock jitter.

Moreover, the nonuniform sampling due to migration of a finite number of uniform samples can be found in [52]. The nonuniform sampling with finite number of samples has been studied in [52, 58, 59]. As for the nonuniform sampling due to data loss, see [60].

4 Sub-Nyquist Sampling

4.1 Sampling Rate Reduction with the Use of Hilbert Transform or Derivative

The required sampling rate of signal reconstruction can be decreased with the proper use of the Hilbert transform or the derivative of a signal. The works of [61] and [62] have reconstructed the original signal by interpolating the samples of both the signal and its Hilbert transform at half the Nyquist rate for the FrFT. Moreover, the work of [62] has proposed a signal reconstruction method by interpolating the samples of both the signal and its first-order derivative at half the Nyquist rate for the FrFT, where the samples of the first-order derivative can be approximated using central difference differentiator. However, the performance of central difference differentiator is highly dependent on the signal-to-noise ratio. For related works for the non-stationary signal reconstruction with the use of the samples of the one-order derivative, see [30]. In 2019, the work of [63] has extended the sampling theorem using first-order derivative to higher-order derivative and proposed a signal reconstruction method by interpolating the samples of the signal and its $1, 2, \cdots, n-1$th-order derivative at $\frac{1}{n}$ Nyquist rate for the FrFT.

4.2 Sampling and Reconstruction of Multiband Signals in the Fractional Fourier Domain

As for the signals with finite disjoint bands in the fractional Fourier domain, uniform sampling is inefficient since the vacant fractional frequency intervals without signal energy can lead to unnecessary high sampling rates. To solve this problem, the work of [64] has proposed a match-pursuit-based signal reconstruction that extends a modulated wideband converter to the fractional Fourier domain by a fractional shift invariant operator. Moreover, the work of [65] has devised a signal reconstruction method in the multiresolution subspaces associated with the fractional wavelet transform, based on which the required sampling rate can be lower than the lowest sampling rate required by uniform sampling.

4.3 Sparse Discrete FrFT

Based on the sampling theorem associated with the FrFT, various types of numerical algorithms have been proposed to compute the discrete

FrFT. Among them, Pei's sampling-type algorithm can achieve relatively low complexity and high precision [66], and therefore it is widely used in applications. However, Pei's sampling-type algorithm will become time-consuming when dealing with large data sets, which necessitates the development of sparse discrete FrFT to reduce the computational complexity [67, 68]. The sparse discrete FrFT proposed in [67] can be used to deal with the signals that are sparsely represented in the fractional Fourier domain and it is more suitable for large data sets [68]. Furthermore, the work of [68] has optimized it by means of exploiting the underlying phase information [67]. The sparse discrete FrFT can be used to the fast parameter estimation approach of linear frequency modulation signal and radar signal processing [67], etc.

5 Future Research Topics

In the future, the following research topics are still worthy to be studied:

1) Sub-Nyquist sampling and spectral analysis of signals with finite disjoint bands in the fractional Fourier domain.

2) Multiple input multiple output sampling [69] and reconstruction of signals that are band-limited or non-bandlimited in the fractional Fourier domain.

3) The FrFT-based sampling theorems with respect to novel sampling schemes such as sampling with finite rate of innovation [70, 71], time encoding machine [72, 73], and one-bit sampling [74], etc.

4) The FrFT-based nonuniform sampling theorem with consideration of multiple factors such as timing offset, clock jitter, and periodic data loss.

6 Conclusion

In this paper, we have reviewed recent developments of the sampling theorems associated with the FrFT. We first have given the results about signal reconstruction and the spectral analysis associated with the FrFT with respect to uniform sampling and nonuniform samplings due to various factors such as clock jitter/random perturbation and timing offset, etc. Then we have reviewed the works about sub-Nyquist sampling associated with the Hilbert transform/derivative and multiband signals, etc. Finally, we have provided several future research topics of the sampling theorem associated with the FrFT.

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