

RESEARCH ARTICLE

Evolution law of Wigner function in laser process

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Based on the density operator's operator-sum representation recently obtained by Fan and Hu for a laser process (Opt. Commun., 2008, 281: 5571; Opt. Commun., 2009, 282: 932; Phys. Lett. B, 2008, 22: 2435), we derive the evolution law of Wigner operator, the law is concisely expressed in the normally ordered form $\Delta(\alpha, \alpha^*, t) = \frac{T}{\pi} : \exp[-2T(a^\dagger e^{-(\kappa-g)t} - \alpha^*)(ae^{-(\kappa-g)t} - \alpha)] :$, where g and κ are the cavity gain and the loss, respectively, and $T \equiv (\kappa - g)(\kappa + g - 2ge^{-2(\kappa-g)t})^{-1}$. When $t = 0$, $\Delta(\alpha, \alpha^*, t) \rightarrow \frac{1}{\pi} : \exp[-2(a^\dagger - \alpha^*)(a - \alpha)] :$, which is the initial Wigner operator. Using this formalism the evolution law of Wigner functions in laser process can be directly obtained.

Keywords Kraus operator, Wigner operator, laser process

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1 Introduction

It is known that the decay (decoherence) of quantum information due to the interaction between a system and its environment can be described by an evolution from the initial density operator ρ_0 of the system to $\rho(t)$ [1, 2],

$$\rho(t) = \sum_{n=0}^{\infty} M_n \rho_0 M_n^\dagger \quad (1)$$

Such an expression is named an operator-sum representation, and M_n is called Kraus operator. In quantum optics theory the mechanism of laser in the lowest-order approximation can be described by the following master equation of density operator [3, 4]

$$\begin{aligned} \frac{d\rho(t)}{dt} = & g[2a^\dagger \rho(t)a - aa^\dagger \rho(t) - \rho(t)aa^\dagger] \\ & + \kappa[2a\rho(t)a^\dagger - a^\dagger a \rho(t) - \rho(t)a^\dagger a] \end{aligned} \quad (2)$$

where g and κ are the cavity gain and the loss, respectively. In the preceding papers [5–7] we have solved this equation and derived its infinitive operator-sum representation with the corresponding Kraus operator. In this work we shall go an important step further, i.e., to demonstrate how an initial Wigner function of density operator $\rho(0)$ evolves into the Wigner function of $\rho(t)$. Wigner function $W_\rho(\alpha, \alpha^*)$ is a quasi-probability repre-

sentation for the density operator, which is related to ρ by

$$W_\rho(\alpha, \alpha^*) = \text{Tr}[\rho \Delta(\alpha, \alpha^*)] \quad (3)$$

where $\Delta(\alpha, \alpha^*)$ is the Wigner operator originally defined as [8]

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} dv e^{ipv} |q + \frac{v}{2}\rangle \langle q - \frac{v}{2}| &= \Delta(q, p) \\ \alpha &= \frac{q + ip}{\sqrt{2}} \end{aligned} \quad (4)$$

here $|q\rangle$ is the coordinate eigenstate, in the Fock space is expressed as

$$|q\rangle = \pi^{-1/4} \exp\left(-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right) |0\rangle \quad (5)$$

$a^\dagger$ is the bosonic creation operator, $[a, a^\dagger] = 1$, $|0\rangle$ is the vacuum state, $a|0\rangle = 0$, so we focus on the time evolution of Wigner operator, $\text{Tr}[\rho(t)\Delta(\alpha, \alpha^*)]$. We can ascribe $\text{Tr}[\rho(t)\Delta(\alpha, \alpha^*)]$ to $\text{Tr}[\rho_0 \Delta(\alpha, \alpha^*, t)]$, where $\Delta(\alpha, \alpha^*, t) = \sum_{n=0}^{\infty} M_n^\dagger \Delta(\alpha, \alpha^*) M_n$, and we shall demonstrate the concise normally ordered form

$$\begin{aligned} \Delta(\alpha, \alpha^*, t) = & \frac{T}{\pi} : \exp\{-2T[a^\dagger e^{-(\kappa-g)t} - \alpha^*] \\ & \times [ae^{-(\kappa-g)t} - \alpha]\} : \end{aligned} \quad (6)$$

here $T \equiv \frac{\kappa-g}{\kappa+g-2ge^{-2(\kappa-g)t}}$. Using this formalism the evolution law of Wigner functions from W_{ρ_0} to $W_{\rho(t)}$ in laser process can be directly obtained. We shall derive this neat equation.

Our work is arranged as follows: In Section 2 we briefly review the operator-sum-representation solution to the laser master equation (2) after introducing the entangled state representation; In Section 3 we prove Eq. (6). In Section 4 we calculate the time evolution of Wigner function for the coherent state and the number state.

2 Briefly review the solution to Eq. (2)

In order to expose our way of obtaining the Kraus operator for Eq. (1), let us introduce the two-mode entangled state [9–12]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right) |0\tilde{0}\rangle \quad (7)$$

where $\tilde{a}^\dagger$ is a fictitious mode independent of the real mode $a^\dagger$, $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $[\tilde{a}, \tilde{a}^\dagger] = 1$. The way of doubling the real mode with use of the fictitious mode is firstly proposed by Umezawa and Takahashi and is named as thermo dynamics [13]. The state $|\eta=0\rangle \equiv |I\rangle$ possesses the properties

$$a|I\rangle = \tilde{a}^\dagger|I\rangle, \quad a^\dagger|I\rangle = \tilde{a}|I\rangle \quad (8)$$

$$(a^\dagger a)^n|I\rangle = (\tilde{a}^\dagger \tilde{a})^n|I\rangle \quad (9)$$

Operating the both sides of (2) on $|I\rangle$ and denoting $|\rho\rangle = \rho|I\rangle$ yields the time-evolution equation for $|\rho(t)\rangle$,

$$\begin{aligned} \frac{d}{dt}|\rho(t)\rangle &= [g(2a^\dagger \tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger) \\ &\quad + \kappa(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a})]|\rho(t)\rangle \end{aligned} \quad (10)$$

whose formal solution is

$$|\rho(t)\rangle = U(t)|\rho_0\rangle, \quad |\rho_0\rangle = \rho_0|I\rangle \quad (11)$$

and

$$\begin{aligned} U(t) &= \exp[gt(2a^\dagger \tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger) \\ &\quad + \kappa t(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a})] \end{aligned} \quad (12)$$

By disentangling the exponential operator $U(t)$ in our previous work [5–7] we have derived

$$\begin{aligned} |\rho(t)\rangle &= T_3 e^{gT_1 a^\dagger \tilde{a}^\dagger} : \exp[(T_2 - 1)(\tilde{a}^\dagger \tilde{a} + a^\dagger a)] : \\ &\quad \times e^{\kappa T_1 a \tilde{a}} |\rho_0\rangle \end{aligned} \quad (13)$$

where

$$\begin{aligned} T_1 &= \frac{1 - e^{-2(\kappa-g)t}}{\kappa - ge^{-2t(\kappa-g)}}, \quad T_2 = \frac{(\kappa - g)e^{-(\kappa-g)t}}{\kappa - ge^{-2t(\kappa-g)}} \\ T_3 &= \frac{\kappa - g}{\kappa - ge^{-2t(\kappa-g)}} = 1 - gT_1 \end{aligned} \quad (14)$$

Further, noticing $\tilde{a}\rho_0 = \rho_0\tilde{a}$ and (9), as well as

$$\begin{aligned} &e^{\tilde{a}^\dagger \tilde{a} \ln T_2} e^{\kappa T_1 a \tilde{a}} |\rho_0\rangle \\ &= e^{\tilde{a}^\dagger \tilde{a} \ln T_2} \sum_{i=0}^{\infty} \frac{\kappa^i T_1^i}{i!} a^i \rho_0 \tilde{a}^i |I\rangle \\ &= \sum_{i=0}^{\infty} \frac{\kappa^i T_1^i}{i!} a^i \rho_0 a^{\dagger i} e^{\tilde{a}^\dagger \tilde{a} \ln T_2} |I\rangle \\ &= \sum_{i=0}^{\infty} \frac{\kappa^i T_1^i}{i!} a^i \rho_0 a^{\dagger i} e^{a^\dagger a \ln T_2} |I\rangle \end{aligned} \quad (15)$$

we have

$$\begin{aligned} |\rho(t)\rangle &= T_3 e^{gT_1 a^\dagger \tilde{a}^\dagger} e^{(\tilde{a}^\dagger \tilde{a} + a^\dagger a) \ln T_2} e^{\kappa T_1 a \tilde{a}} |\rho_0\rangle \\ &= T_3 \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!} a^{\dagger j} e^{a^\dagger a \ln T_2} a^i \rho_0 a^{\dagger i} e^{a^\dagger a \ln T_2} a^j |I\rangle \\ &= \sum_{i,j=0}^{\infty} T_3 \frac{\kappa^i g^j T_1^{i+j}}{i!j!} a^{\dagger j} e^{a^\dagger a \ln T_2} a^i \rho_0 a^{\dagger i} \\ &\quad \times e^{a^\dagger a \ln T_2} a^j |I\rangle \\ &= \sum_{i,j=0}^{\infty} T_3 \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \rho_0 a^{\dagger i} a^j \\ &\quad \times e^{a^\dagger a \ln T_2} |I\rangle \end{aligned} \quad (16)$$

Comparing this with the standard form of the operator-sum representation in (1), we see

$$\rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger \quad (17)$$

thus the Kraus operator for laser process is

$$M_{ij} = \sqrt{\frac{\kappa^i g^j T_3 T_1^{i+j}}{i!j!T_2^{2j-1}}} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \quad (18)$$

and we can prove the unitarity

$$\sum_{i,j=0}^{\infty} M_{ij}^\dagger M_{ij} = 1 \quad (19)$$

3 The proof of Eq. (6)

Using the IWOP [14–17] and $|0\rangle\langle 0| =: \exp[-a^\dagger a] : we can prove that$

$$\begin{aligned} \Delta(q, p) &= \frac{1}{2\pi\sqrt{\pi}} \int_{-\infty}^{\infty} dv e^{ipv} \\ &\quad \times : \exp \left[-\frac{\left(q + \frac{v}{2}\right)^2}{2} + \sqrt{2} \left(q + \frac{v}{2}\right) a^\dagger - \frac{a^{\dagger 2}}{2} \right. \\ &\quad \left. - a^\dagger a - \frac{\left(q - \frac{v}{2}\right)^2}{2} + \sqrt{2} \left(q - \frac{v}{2}\right) a - \frac{a^2}{2} \right] : \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} : e^{-(q-Q)^2 - (p-P)^2} : \\
&= \frac{1}{\pi} : e^{-2(a^\dagger - \alpha^*)(a - \alpha)} :
\end{aligned} \quad (20)$$

where

$$\alpha = \frac{q + ip}{\sqrt{2}}, \quad a = \frac{Q + iP}{\sqrt{2}} \quad (21)$$

The coherent state representation of $\Delta(q, p)$ is

$$\Delta(q, p) \equiv \Delta(\alpha, \alpha^*) = \int \frac{d^2 z}{\pi^2} |\alpha + z\rangle \langle \alpha - z| e^{\alpha z^* - \alpha^* z} \quad (22)$$

where [18, 19]

$$|z\rangle = e^{-|z|^2/2 + za^\dagger} |0\rangle \quad (23)$$

since performing this integration yields

$$\begin{aligned}
\Delta(\alpha, \alpha^*) &= \frac{1}{\pi} : e^{-2(a^\dagger - \alpha^*)(a - \alpha)} : \\
&= \frac{1}{\pi} e^{2\alpha a^\dagger} e^{a^\dagger a \ln(-1)} \\
&\quad \times e^{2\alpha^* a} e^{-2\alpha \alpha^*}
\end{aligned} \quad (24)$$

where we have used the operator identity

$$e^{\lambda a^\dagger a} = : \exp[(e^\lambda - 1)a^\dagger a] : \quad (25)$$

and

$$: e^{-2a^\dagger a} := (-1)^{a^\dagger a} = e^{a^\dagger a \ln(-1)} = (-1)^N \quad (26)$$

Due to (17), the Wigner function for $\rho(t)$ is equal to

$$\begin{aligned}
W_{\rho(t)}(p, q) &= \text{Tr}[\rho(t)\Delta(\alpha, \alpha^*)] \\
&= \text{Tr} \left[\sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger \Delta(\alpha, \alpha^*) \right] \\
&= \text{Tr} \left[\sum_{i,j=0}^{\infty} \rho_0 M_{ij}^\dagger \Delta(\alpha, \alpha^*) M_{ij} \right] \\
&= \text{Tr}[\rho_0 \Delta(\alpha, \alpha^*, t)]
\end{aligned} \quad (27)$$

where we introduced

$$\Delta(\alpha, \alpha^*, t) = \sum_{i,j=0}^{\infty} M_{ij}^\dagger \Delta(\alpha, \alpha^*) M_{ij} \quad (28)$$

Thus our task becomes to calculate $\Delta(\alpha, \alpha^*, t)$, by using (17), (18), (20), (24) and

$$(-1)^N a = -a(-1)^N \quad (29)$$

$$e^{a^\dagger a \ln A} f(a^\dagger) e^{-a^\dagger a \ln A} = f(Aa^\dagger)$$

$$e^{a^\dagger a \ln A} f(a) e^{-a^\dagger a \ln A} = f(a/A) \quad (30)$$

we have

$$\begin{aligned}
\Delta(\alpha, \alpha^*, t) &= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} a^{\dagger i} a^j e^{a^\dagger a \ln T_2} e^{2\alpha a^\dagger} \\
&\quad \times e^{a^\dagger a \ln(-1)} e^{2\alpha^* a} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \\
&= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} a^{\dagger i} a^j e^{2\alpha T_2 a^\dagger} \\
&\quad \times e^{a^\dagger a \ln T_2} (-1)^N e^{2\alpha^* a} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \\
&= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} a^{\dagger i} a^j e^{2\alpha T_2 a^\dagger} \\
&\quad \times e^{-2\alpha^* a/T_2} e^{a^\dagger a \ln(-T_2^2)} a^{\dagger j} a^i \\
&= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} a^{\dagger i} a^j e^{2\alpha T_2 a^\dagger} \\
&\quad \times e^{-2\alpha^* a/T_2} a^{\dagger j} a^i e^{a^\dagger a \ln(-T_2^2)} (-T_2^2)^{j-i}
\end{aligned} \quad (31)$$

Then using

$$e^{Aa} f(a^\dagger) e^{-Aa} = \sum_{n=0}^{\infty} \frac{A^n}{n!} f^{(n)}(a^\dagger) = f(a^\dagger + A) \quad (32)$$

$$e^{Aa^\dagger} f(a) e^{-Aa^\dagger} = \sum_{n=0}^{\infty} \frac{A^n}{n!} (-1)^n f^{(n)}(a^\dagger) = f(a - A) \quad (33)$$

we further convert (31) to

$$\begin{aligned}
\Delta(\alpha, \alpha^*, t) &= \frac{\exp(-2\alpha^* \alpha T_3)}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} a^{\dagger i} e^{2\alpha T_2 a^\dagger} \\
&\quad \times (a + 2\alpha T_2)^j (a^\dagger - 2\alpha^*/T_2)^j e^{-2\alpha^* a/T_2} a^i \\
&\quad \times e^{a^\dagger a \ln(-T_2^2)} (-T_2^2)^{j-i}
\end{aligned} \quad (34)$$

Then using the completeness relation of the coherent state

$$\begin{aligned}
\int \frac{d^2 z}{\pi} |z\rangle \langle z| &= \int \frac{d^2 z}{\pi} : \exp[-|z|^2 + za^\dagger + z^* a - a^\dagger a] : \\
&= 1
\end{aligned} \quad (35)$$

and $a|z\rangle = z|z\rangle$ we obtain

$$\begin{aligned}
\Delta(\alpha, \alpha^*, t) &= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i,j=0}^{\infty} \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} e^{2\alpha T_2 a^\dagger} a^{\dagger i} \\
&\quad \times \int \frac{d^2 z}{\pi} (z + 2\alpha T_2)^j |z\rangle \langle z| (z^* - 2\alpha^*/T_2)^j \\
&\quad \times a^i e^{-2\alpha^* a/T_2} e^{a^\dagger a \ln(-T_2^2)} (-T_2^2)^{j-i} \\
&= \frac{e^{-2\alpha^* \alpha T_3}}{\pi} \sum_{i=0}^{\infty} \frac{\kappa^i T_1^i}{i!} e^{2\alpha T_2 a^\dagger} a^{\dagger i} \int \frac{d^2 z}{\pi}
\end{aligned}$$

$$\begin{aligned} & \times e^{-gT_1(z+2\alpha T_2)(z^*-2\alpha^*/T_2)} \\ & \times |z\rangle\langle z| a^i e^{-2\alpha^* a/T_2} e^{a^\dagger a \ln(-T_2^2)} (-T_2^2)^{-i} \end{aligned} \quad (36)$$

Then using the IWOP technique

$$\begin{aligned} \Delta(\alpha, \alpha^*, t) &= \frac{e^{-2\alpha^* \alpha T_3}}{\pi(1+gT_1)} \sum_{i=0}^{\infty} \frac{\kappa^i T_1^i}{i!} (-T_2^2)^{-i} e^{2\alpha T_2 a^\dagger} \\ & \times : a^{\dagger i} \exp \left[\frac{gT_1(2\alpha^*/T_2 a - a^\dagger a - 2\alpha T_2 a^\dagger + 4\alpha^* \alpha)}{1+gT_1} \right] a^i : \\ & \times e^{-2\alpha^* a/T_2} e^{a^\dagger a \ln(-T_2^2)} \\ &= \frac{e^{\frac{gT_1-1}{1+gT_1} 2\alpha^* \alpha T_3}}{\pi(1+gT_1)} e^{2\alpha T_2 a^\dagger} : \exp \left[\frac{gT_1}{1+gT_1} (2\alpha^*/T_2 a \right. \\ & \quad \left. - a^\dagger a - 2\alpha T_2 a^\dagger) - \frac{\kappa T_1}{T_2^2} a^\dagger a \right] : \\ & \times e^{-2\alpha^* a/T_2} e^{a^\dagger a \ln(-T_2^2)} \\ &= \frac{e^{\frac{gT_1-1}{1+gT_1} 2\alpha^* \alpha T_3}}{\pi(1+gT_1)} e^{\frac{2\alpha T_2 a^\dagger}{1+gT_1}} \\ & \times : \exp \left[\left(\frac{1}{1+gT_1} - \frac{\kappa T_1}{T_2^2} - 1 \right) a^\dagger a \right] : \\ & \times e^{-\frac{2\alpha^* a}{T_2(1+gT_1)}} e^{a^\dagger a \ln(-T_2^2)} \\ &= \frac{e^{\frac{gT_1-1}{1+gT_1} 2\alpha^* \alpha T_3}}{\pi(1+gT_1)} e^{\frac{2\alpha T_2 a^\dagger}{1+gT_1}} \exp \left[\ln \left(\frac{1}{1+gT_1} - \frac{\kappa T_1}{T_2^2} \right) a^\dagger a \right] \\ & \times e^{a^\dagger a \ln(-T_2^2)} e^{\frac{2\alpha^* T_2 a}{1+gT_1}} \\ &= \frac{e^{\frac{gT_1-1}{1+gT_1} 2\alpha^* \alpha T_3}}{\pi(1+gT_1)} e^{\frac{2\alpha T_2 a^\dagger}{1+gT_1}} \exp \left[\ln \left(\kappa T_1 - \frac{T_2^2}{1+gT_1} \right) a^\dagger a \right] \\ & \times e^{\frac{2\alpha^* T_2 a}{1+gT_1}} \end{aligned} \quad (37)$$

By introducing

$$T \equiv \frac{\kappa - g}{\kappa + g - 2ge^{-2(\kappa-g)t}} \quad (38)$$

then using (14) we see

$$\frac{T_2}{1+gT_1} = Te^{-(\kappa-g)t}$$

so Eq. (37) and can simplified as

$$\begin{aligned} \Delta(\alpha, \alpha^*, t) &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} e^{2\alpha T e^{-(\kappa-g)t} a^\dagger} \\ & \times e^{\ln(1-2Te^{-2(\kappa-g)t}) a^\dagger a} e^{2\alpha^* T e^{-(\kappa-g)t} a} \\ &= \frac{T}{\pi} : e^{-2T(a^\dagger e^{-(\kappa-g)t} - \alpha^*)(ae^{-(\kappa-g)t} - \alpha)} : \end{aligned} \quad (39)$$

This is the general formula about the time evolution of the Wigner operator in a laser process, it is elegant, using it the various Wigner functions' evolution can be directly obtained. When $t = 0$, $\Delta(\alpha, \alpha^*, t) \rightarrow \frac{1}{\pi} : \exp[-2(a^\dagger - \alpha^*)(a - \alpha)] :$, which is the initial Wigner

operator, as expected.

4 Calculating the time evolution of Wigner function for coherent state and number state

According to (27) the Wigner function of $\rho(t)$ can be calculated by $W_{\rho(t)}(p, q) = \text{Tr}[\rho_0 \Delta(\alpha, \alpha^*, t)]$. When the initial state is a pure number state $|n\rangle\langle n|$, the initial Wigner function is

$$\langle n | \Delta(\alpha, \alpha^*) | n \rangle = \frac{1}{\pi} e^{-2|\alpha|^2} L_n(4|\alpha|^2) (-1)^n \quad (40)$$

where $L_n(x)$ is the Laguerre polynomial,

$$L_n(x) = \sum_{l=0}^n \binom{n}{l} \frac{(-x)^l}{l!} \quad (41)$$

then using $a|n\rangle = \sqrt{n}|n-1\rangle$,

$$\begin{aligned} e^{\lambda a} |n\rangle &= \sum_{l=0}^n \frac{(\lambda a)^l}{l!} |n\rangle = \sum_{l=0}^n \frac{\lambda^l}{l!} \sqrt{\frac{n!}{(n-l)!}} |n-l\rangle \\ \lambda &= 2Te^{-(\kappa-g)t} \alpha^* \end{aligned} \quad (42)$$

we have

$$\begin{aligned} \langle n | \Delta(\alpha, \alpha^*, t) | n \rangle &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} \sum_{m=0, l=0}^n \frac{\lambda^l \lambda^{*m}}{l! m!} \sqrt{\frac{n! n!}{(n-m)! (n-l)!}} \\ & \times \langle n-m | e^{a^\dagger a \ln(1-2Te^{-2(\kappa-g)t})} | n-l \rangle \\ &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} \sum_{m=0, l=0}^n \frac{\lambda^l \lambda^{*m}}{l! m!} [1 - 2Te^{-2(\kappa-g)t}]^{n-l} \\ & \times \sqrt{\frac{n! n!}{(n-m)! (n-l)!}} \delta_{m, l} \\ &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} \sum_{l=0}^n \frac{n! |\lambda|^{2l}}{(l!)^2 (n-l)!} (1 - 2Te^{-2(\kappa-g)t})^{n-l} \\ &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} [1 - 2Te^{-2(\kappa-g)t}]^n L_n \left(\frac{|\lambda|^2}{2Te^{-2(\kappa-g)t} - 1} \right) \\ &= \frac{e^{-2T\alpha^* \alpha T}}{\pi} [1 - 2Te^{-2(\kappa-g)t}]^n L_n \left(\frac{4T^2 e^{-2(\kappa-g)t} |\alpha|^2}{2Te^{-2(\kappa-g)t} - 1} \right) \end{aligned} \quad (43)$$

when $t = 0$, $T = 1$, Eq. (43) reduces to (40), as expected. When the initial state is a pure coherent state $\rho_0 = |z\rangle\langle z|$, then using (39) the final state's Wigner function is

$$\begin{aligned} W_{\rho(t)} &= \frac{T}{\pi} \langle z | : \exp \{ -2T(a^\dagger e^{-(\kappa-g)t} - \alpha^*) \\ & \quad \times [ae^{-(\kappa-g)t} - \alpha] \} : | z \rangle \\ &= \frac{T}{\pi} \exp[-2T(z^* e^{-(\kappa-g)t} - \alpha^*)(ze^{-(\kappa-g)t} - \alpha)] \end{aligned} \quad (44)$$

In general cases, we can write ρ_0 as $\mathcal{P}$ -representation

$$\rho_0 = \int d^2z \mathcal{P}(z) |z\rangle\langle z| \quad (45)$$

then

$$\begin{aligned} W_{\rho(t)} &= \text{Tr}[\rho_0 \Delta(\alpha, \alpha^*, t)] \\ &= \int d^2z \mathcal{P}(z) \langle z | \frac{T}{\pi} : \exp\{-2T(a^\dagger e^{-(\kappa-g)t} - \alpha^*) \\ &\quad \times [ae^{-(\kappa-g)t} - \alpha]\} : | z \rangle \\ &= \frac{T}{\pi} \int d^2z \mathcal{P}(z) \exp\{-2T(z^* e^{-(\kappa-g)t} - \alpha^*) \\ &\quad \times [ze^{-(\kappa-g)t} - \alpha]\} \end{aligned} \quad (46)$$

5 Summary

In summary, for a laser process we have derived a concise formula $\text{Tr}[\rho_0 \Delta(\alpha, \alpha^*, t)]$ for the time evolution law of Wigner function $\text{Tr}[\rho(t) \Delta(\alpha, \alpha^*)]$, and $\Delta(\alpha, \alpha^*, t)$ possesses a concise form. Using it various Wigner functions' evolution can be directly obtained. Finally, we point out that the approach for Wigner function's evolution in this paper is more direct than the approach taken in Ref. [21]. For advanced tomography theory related to Wigner operator we refer to Ref. [21].

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