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具 p-Laplace 算子的分数阶脉冲微分方程奇异边值问题的解

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摘要：用 Banach 压缩映像原理和 Krsnoasel'skii 不动点定理证明一类具有 p-Laplace 算子的分数阶脉冲微分方程奇异边值问题解的唯一性和存在性.

关键词：分数阶微分方程；脉冲；不动点定理；奇异边值问题；p-Laplace 算子

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Solutions of Singular Boundary Value Problems for Fractional Impulsive Differential Equations with p-Laplacian Operator

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Abstract: We proved the uniqueness and existence of solutions for a class of singular boundary value problems of fractional impulsive differential equations with p-Laplacian operators by using Banach contraction mapping principle and Krsnoasel'skii fixed point theorem.

Keywords: fractional differential equation; impulse; fixed point theorem; singular boundary value problem; p-Laplacian operator

0 引 言

分数阶微分方程广泛应用于空气动力学、复杂介质电动力学、聚合物流变学、实验数据的拟合等领域，目前，关于分数阶微分方程边值问题解的存在性和唯一性研究已有很多结果[1-14]. 脉冲微分方程在物理学、社会学和信号处理建模中具有重要作用[1-6]，它可以描述模型在某些时刻发生变化的状态，不能由经典的微分方程建模表示. 关于分数阶脉冲微分方程奇异边值问题的研究已得到广泛关

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注[7-8]. 近年来，具 p-Laplace 算子的分数阶微分方程的数学模型在多孔介质中的湍流过滤、血液流动、流变学和材料科学中应用广泛，因此研究具 p-Laplace 算子的分数阶微分方程边值问题备受关注[9-14].

文献[1]研究了分数阶脉冲微分方程边值问题

$$\begin{cases} D_{0+}^{\beta}\varphi_p(D_{0+}^{\beta}u(t))=f(t,u(t)), & t\in J', \\ \Delta u(t_k)=I_k(u(t_k)), \quad \Delta\varphi_p(D_{0+}^{\alpha}u(t_k))=b_k, & k=1,2,\cdots,m, \\ u(0)=u_0, \quad D_{0+}^{\alpha}u(0)=u_1 \end{cases}$$

解的存在性，其中 $0<\alpha,\beta\leqslant 1$，$1<\alpha+\beta\leqslant 2$，$0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1$，$J'=[0,1]\backslash\{t_1,t_2,\cdots,t_m\}$，$f\in C([0,1]\times\mathbb{R},\mathbb{R})$，$u_0,u_1\in\mathbb{R}$，$b_k\in\mathbb{R}$，$k=1,2,\cdots,m$，$I_k\in C(\mathbb{R},\mathbb{R})$. 文献[2]用 Schauder 不动点定理、不动点指数定理研究了分数阶脉冲微分方程积分边值问题

$$\begin{cases} -D_{0+}^{\alpha}u(t)=\lambda a(t)f(t,u,u'), & t\in(0,1)\backslash\{t_k\}_{k=1}^{m}, \\ \Delta u(t_k)=I_k(u(t_k)), & t=t_k, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0), & u'(1)=\int_0^1 u(s)\mathrm{d}H(s) \end{cases}$$

的解，其中 $n-1<\alpha\leqslant n$. D_{0+}^{α} 是 Riemann-Liouville 分数阶导数，$a(t)\in C((0,1),[0,+\infty))$，$f\in C([0,1]\times[0,+\infty),[0,+\infty))$，$I_k\in C(\mathbb{R}^+,\mathbb{R}^+)$. 文献[8]研究了分数阶微分方程奇异边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t)), & 0<t<1, \\ u(0)=u'(t)=u''(t)=u''(1)=0 \end{cases}$$

正解的唯一性，其中 $3<\alpha\leqslant 4$，D_{0+}^{α} 为标准的 Riemann-Liouville 分数阶微分，$f:(0,1]\times[0,+\infty)\to[0,+\infty)$，$f$ 在 $t=0$ 处奇异.

目前，对具 p-Laplace 算子的分数阶微分方程边值问题的研究已有很多结果，但对具 p-Laplace 算子的分数阶脉冲微分方程奇异边值问题的研究报道较少，基于此，本文主要研究具 p-Laplace 算子的分数阶脉冲微分方程奇异边值问题

$$\begin{cases} {}^{\mathrm{C}}D_{0+}^{\beta}\varphi({}^{\mathrm{C}}D_{0+}^{\alpha}u(t))=\lambda f(t,u(t),u'(t)), & t\in J', \quad \lambda>0, \\ \Delta u(t_k)=I_k(u(t_k)), \quad \Delta u'(t_k)=Q_k(u(t_k),u'(t_k)), & k=1,2,\cdots,m, \\ u(0)=u_0, \quad u(1)=u_1, \quad {}^{\mathrm{C}}D_{0+}^{\alpha}u(0)=0 \end{cases} \tag{1}$$

解的存在性，其中：${}^{\mathrm{C}}D_{0+}^{\alpha}$，${}^{\mathrm{C}}D_{0+}^{\beta}$ 为标准的 Caputo 分数阶导数，$0<\beta\leqslant 1<\alpha\leqslant 2$，且 $\beta+1<\alpha$；$0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1$，$J'=[0,1]\backslash\{t_1,t_2,\cdots,t_m\}$；$f\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$，非线性函数 f 可能在 $t=0$ 处具有奇性；$I_k(u)\in C(\mathbb{R},\mathbb{R})$，$Q_k(u,u')\in C(\mathbb{R}^2,\mathbb{R})$，$\Delta u=u(t_k^+)-u(t_k^-)$，$\Delta u'=u'(t_k^+)-u'(t_k^-)$，$\Delta u(t_k^+)$，$\Delta u(t_k^-)$ 和 $\Delta u'(t_k^+)$，$\Delta u'(t_k^-)$ 分别为 $\Delta u(t_k)$ 和 $\Delta u'(t_k)$ 在 $t=t_k$ 处的左右极限；$\varphi_p(s)=|s|^{p-2}s$，$p>1$，$(\varphi_p)^{-1}(s)=\varphi_q(s)$，且 $1/p+1/q=1$.

1　预备知识

为讨论方程解的存在性和唯一性，引入以下空间：令 $J=[0,1]$，$0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$，$J_0=[t_0,t_1]$，$\cdots$，$J_m=(t_m,t_{m+1}]$，$PC(J,\mathbb{R})=\{u:J\to\mathbb{R}\mid u\in C(J_k),\ k=0,1,\cdots,m$ 且 $u(t_k^+)$ 存在$\}$，其范数为 $\|u\|_{PC}=\sup\limits_{t\in J}|u(t)|$，并有 $PC'(J,\mathbb{R})=\{u:J\to\mathbb{R}\mid u\in C(J_k),\ k=0,1,\cdots,m$ 且 $u'(t_k^+)$ 存在$\}$，其范数为 $\|u\|_{PC'}=\sup\limits_{t\in J}|u(t)|+\sup\limits_{t\in J}|u'(t)|$，显然 $PC(J,\mathbb{R})$ 与 $PC'(J,\mathbb{R})$ 为 Banach 空间.

定义 1[7]　函数 $f:[0,+\infty)\to\mathbb{R}$ 的 $\alpha>0$ 阶积分定义为

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\mathrm{d}s,$$

其中右边是在 $[0,+\infty)$ 上逐点定义的.

定义 2[7]　函数 $f:[0,+\infty)\to\mathbb{R}$ 的 $\alpha>0$ 阶 Caputo 微分定义为

$${}^{\mathrm{C}}D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s,$$

其中 $n=[\alpha]+1$，右边是在 $[0,+\infty)$ 上逐点定义的.

定理 1(压缩映像原理)[7] 设(X,d)为非空完备度量空间，$T:X\to X$为X上的压缩映射，存在$q<1$，使得$\forall x,y\in X$，都有$d(Tx,Ty)<qd(x,y)$，则存在唯一的$\xi\in X$，使得$T\xi=\xi$.

定理 2(Krsnoasel'skii 不动点定理)[7] 令M是 Banach 空间X中的非空闭凸子集，并且A,B为算子，使得：

1) 对$\forall x,y\in M$，$Ax+By\in M$；

2) A为全连续算子；

3) B为压缩映射.

则存在$z\in M$，使得$z=Az+Bz$.

引理 1(Hölder 不等式)[7] 假设$\gamma,\delta\geqslant 1$，满足$1/\gamma+1/\delta=1$，若$a\in L^{\gamma}(J)$，$b\in L^{\delta}(J)$对于$1\leqslant\delta\leqslant\infty$，$ab\in L^{1}(J)$，则有$\|ab\|_{L^{1}(J)}\leqslant\|a\|_{L^{\gamma}(J)}\|b\|_{L^{\delta}(J)}$.

引理 2[2] 若$\alpha>0$，则有：

1) $I_{0+}^{\alpha}({}^{\mathrm{C}}D_{0+}^{\alpha}u(t))=u(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1}$；

2) ${}^{\mathrm{C}}D_{0+}^{\alpha}(I_{0+}^{\alpha}u(t))=u(t)$.

其中$c_i\in\mathbb{R}$，$i=0,1,\cdots,n-1$，$n=[\alpha]+1$.

引理 3[4] 对p-Laplace 算子$\varphi_p(s)=|s|^{p-2}s(p>1)$，下列结论成立：

1) 如果$1<p<2$，$uv>0$，$|u|,|v|\geqslant r>0$，则有$|\varphi_p(u)-\varphi_p(v)|\leqslant(p-1)r^{p-2}|u-v|$；

2) 如果$p>2$，$|u|,|v|<R$，则有$|\varphi_p(u)-\varphi_p(v)|\leqslant(p-1)R^{p-2}|u-v|$.

引理 4 对给定的函数$y\in C[0,1]$，$u(t)$是如下分数阶脉冲微分方程积分边值问题

$$\begin{cases}{}^{\mathrm{C}}D_{0+}^{\beta}\varphi({}^{\mathrm{C}}D_{0+}^{\alpha}u(t))=y(t),\quad t\in J,\quad \lambda>0,\\ \Delta u(t_k)=I_k(u(t_k)),\quad \Delta u'(t_k)=Q_k(u(t_k),u'(t_k)),\quad k=1,2,\cdots,m,\\ u(0)=u_0,\quad u(1)=u_1,\quad {}^{\mathrm{C}}D_{0+}^{\alpha}u(0)=0\end{cases}\tag{2}$$

的解，当且仅当有如下形式：

$$u(t)=\begin{cases}\displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_1-c_2t,\quad t\in J_0,\\ \displaystyle\int_{t_k}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\ \quad\displaystyle\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\ \quad\displaystyle\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\ \quad\displaystyle\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)Q_i(u,u')+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u,u')-c_1-c_2t,\quad t\in J_k,\end{cases}\tag{3}$$

其中$c_1=-u_0$，

$$\begin{aligned}c_2=&\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\ &\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\ &\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\sum_{i=1}^{m}I_i(u(t_i))+\\ &\sum_{i=1}^{m}(1-t_m)Q_i(u,u')+\sum_{i=1}^{m-1}(t_m-t_i)Q_i(u,u')+u_0-u_1.\end{aligned}$$

证明：设$u(t)$为问题(1)的解，当$t\in J_0=[0,t_1]$时，由引理 2 中 1)可得

$$\varphi_p({}^{\mathrm{C}}D_{0+}^{\beta}u(t))=I_{0+}^{\beta}y(t)-c_0=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)\mathrm{d}s-c_0,$$

由边值条件${}^{\mathrm{C}}D_{0+}^{\alpha}u(0)=0$ 得 $c_0=0$，因此

$$\varphi_p({}^{\mathrm{C}}D_{0+}^{\beta}u(t))=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)\mathrm{d}s. \tag{4}$$

对式(4)两边取 φ_p^{-1}，再结合引理 2 可得

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_1-c_2t,$$

从而当 $t\in J_0=[0,t_1]$时，有

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_2,$$

若 $t\in J_1=(0,t_1]$，则有

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-d_1-d_2(t-t_1),$$

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t(t-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-d_2,$$

其中 $d_1,d_2\in\mathbb{R}$，从而可得

$$u(t_1^-)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_1-c_2t_1,\qquad u(t_1^+)=-d_1,$$

$$u'(t_1^-)=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_2,\qquad u'(t_1^+)=-d_2.$$

由于 $\Delta u(t_k)=I_k(u(t_k))$，$\Delta u'(t_k)=Q_k(u(t_k),u'(t_k))$，

$$-d_1=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_1-c_2t_1+I_1(u(t_1)),$$

$$-d_2=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s-c_2+Q_1(u(t_1),u'(t_1)),$$

则当 $t\in J_1=(t_1,t_2]$时，有

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&\frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&I_1(u(t_1))+(t-t_1)Q_1(u(t_1),u'(t_1))-c_1-c_2t.\end{aligned}$$

利用数学归纳法，当 $t\in J_k=(t_k,t_{k+1}]$时，有

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^k\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\sum_{i=1}^kI_i(u(t_i))+\\&\sum_{i=1}^k(t-t_i)Q_i(u(t_i),u'(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i),u'(t_i))-c_1-c_2t.\end{aligned}$$

由边值条件 $u(0)=u_0$，$u(1)=u_1$ 可得式(3)；反之，若 $u(t)$为式(3)的形式，由引理 2 中 2)可得 $u(t)$为边值问题(2)的解.

2　主要结果

定义算子 T：$PC'(J,\mathbb{R})\to PC'(J,\mathbb{R})$为

$$(Tu)(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\sum_{i=1}^{k}I_i(u(t_i))+$$
$$\sum_{i=1}^{k}(t-t_i)Q_i(u(t_i),u'(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i),u'(t_i))-c_1-c_2t,$$
$$(Tu)'(t)=\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\sum_{i=1}^{k}Q_i(u(t_i),u'(t_i))-c_2,$$

其中 $c_1=-u_0$,

$$c_2=\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\right)\mathrm{d}s+$$
$$\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+$$
$$\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\sum_{i=1}^{m}I_i(u(t_i))+$$
$$\sum_{i=1}^{m}(1-t_m)Q_i(u(t_i),u'(t_i))+\sum_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i),u'(t_i))+u_0-u_1.$$

假设:

(H_1) 存在正数 N 及非负值函数 $a(t),b(t)\in C[0,1]$, 使得

$$|f(t,u,u')-f(t,v,v')|\leqslant a(t)|u-v|+b(t)|u'-v'|,$$

并且 $A=\sup\limits_{0\leqslant t\leqslant 1}|a(t)|$, $B=\sup\limits_{0\leqslant t\leqslant 1}|b(t)|$, $N^{1/(q-1)}=\sup\limits_{(t,u,u')\in[0,1]\times[0,r]\times[-r,r]}|f(t,u,u')|$;

(H_2) 存在正数 L_1,L_2,L_3,L_I,L_Q, $\forall x,y\in\mathbb{R}$, $x',y'\in\mathbb{R}$, 使得

$$|I_k(x)-I_k(y)|\leqslant L_1|x-y|,\qquad |I_k(\cdot)|\leqslant L_I,$$
$$|Q_k(x,x')-Q_k(y,y')|\leqslant L_2|x-y|+L_3|x'-y'|,\qquad |Q_k(\cdot,\cdot)|\leqslant L_Q;$$

(H_3) 存在 $\mu\in L^{1/\sigma}(J,\mathbb{R})$, $\sigma\in(0,\alpha-1)$, 使得对 $(t,x,x')\in(J\times\mathbb{R}\times\mathbb{R})$, 有 $f(t,x,x')\leqslant\mu(t)$ 成立, 并且有

$$\rho=2mL_1+(4m-2)L_Q+|u_0|+|u_1|<1. \tag{5}$$

定理 3 假设 $1<p<2$, 条件(H_1)~(H_3)成立, $f\in C(J\times\mathbb{R}\times\mathbb{R})$, 则算子 $T:PC'\to PC'$ 为全连续算子.

证明: 由 f,I_i,Q_i 的连续性可知 T 是连续的, 这里不妨设 Ω 为 $PC'(J,\mathbb{R})$ 上的有界子集, 根据 p-Laplace算子的定义和性质并结合条件(H_1), 可得 $|c_1|\leqslant|u_0|$,

$$|c_2|\leqslant\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\right)\right|\mathrm{d}s+$$
$$\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\right)\right|\mathrm{d}s+$$

$$\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+\sum_{i=1}^{m}|I_i(u(t_i))|+$$

$$\sum_{i=1}^{m}|Q_i(u(t_i),u'(t_i))|+\sum_{i=1}^{m-1}|Q_i(u(t_i),u'(t_i))|+|u_0|+|u_1|,$$

从而可得

$$|(Tu)(t)|=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+$$

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+$$

$$\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+\sum_{i=1}^{k}|I_i(u(t_i))|+$$

$$\sum_{i=1}^{k}|Q_i(u(t_i),u'(t_i))|+\sum_{i=1}^{k-1}|Q_i(u(t_i),u'(t_i))|+|c_1|+|c_2|\leqslant$$

$$N\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}+2mL_1+2(2m-1)L_Q+2|u_0|+|u_1|:=L_1,$$

$$|(Tu)'(t)|\leqslant\int_{t_k}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+$$

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\right|\mathrm{d}s+$$

$$\sum_{i=1}^{k}|Q_i(u(t_i),u'(t_i))|+|c_2|\leqslant$$

$$N\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}\frac{m+1}{\Gamma(\alpha)}+mL_1+(3m-1)L_Q+|u_0|+|u_1|:=L_2,$$

于是有 $|(Tu)(t)|+|(Tu)'(t)|\leqslant L_1+L_2:=L$，即算子 T 一致有界.

另一方面，对 $t_1,t_2\in J_k(k=1,2,\cdots,m)$，不妨设 $t_1\leqslant t_2$，可得

$$|(Tu)(t_2)-(Tu)(t_1)|\leqslant\left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_2}(t_2-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s-\right.$$

$$\left.\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_1}(t_1-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s\right|+$$

$$\left|\sum_{i=1}^{k}\frac{t_2-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s-\right.$$

$$\left.\sum_{i=1}^{k}\frac{t_1-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s\right|+$$

$$\left|\sum_{i=1}^{k}(t_2-t_k)Q_i(u(t_i),u'(t_i))-\sum_{i=1}^{k}(t_1-t_k)Q_i(u(t_i),u'(t_i))\right|+(t_2-t_1)|c_2|\leqslant$$

$$\frac{N}{\Gamma(\alpha+1)}\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}((t_2-t_k)^{\alpha}-(t_1-t_k)^{\alpha})+$$

$$\left(\frac{N}{\Gamma(\alpha)}\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}+mL_Q+|c_2|\right)(t_2-t_1),$$

当 $t_2\to t_1$ 时，有 $|(Tu)(t_2)-(Tu)(t_1)|\to 0$，

$$|(Tu)'(t_2)-(Tu)'(t_1)|\leqslant\left|\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t_2}(t_2-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s-\right.$$

$$\left.\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t_1}(t_1-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s\right|\leqslant$$

$$\frac{N}{\Gamma(\alpha)}\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}((t_2-t_k)^{\alpha}-(t_1-t_k)^{\alpha}),$$

当 $t_1\to t_2$ 时，有 $|(Tu)'(t_2)-(Tu)'(t_1)|\to 0$.

因此对 $\forall\varepsilon>0$，总存在不依赖于 t_1,t_2 的 $\delta=\delta(\varepsilon)>0$，使得当 $|t_2-t_1|<\delta$ 时，有 $\|(Tu)(t_2)-(Tu)(t_1)\|<\varepsilon$，从而算子 T 在$[0,1]$上等度连续，由 Arzela-Ascoli 定理知，$T:PC'\to PC'$为全连续算子.

定理 4 假设 $1<p<2$，条件$(H_1)\sim(H_3)$成立，则边值问题(1)至少存在一个解.

证明：选取

$$r>M\|\mu\|_{L^{\lambda/\sigma}}^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)+2mL_I+(4m-2)L_Q+|u_0|+|u_1|,$$

其中 $M=\left(\frac{1}{\Gamma(\beta)}\left(\frac{1-\sigma}{\beta-\sigma}\right)^{1-\sigma}\right)^{q-1}$. 令 $B_r=\{u\in PC'(J,\mathbb{R})\mid\|u\|\leqslant r\}$，定义算子 P,Q 为

$$\begin{aligned}(Pu)(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\\&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s+\\&\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}(Qu)(t)=&\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}(t-t_i)Q_i(u(t_i),u'(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i),u'(t_i))-\\&\sum_{i=1}^{m}tI_i(u(t_i))-\sum_{i=1}^{m}t(1-t_m)Q_i(u(t_i),u'(t_i))+\\&\sum_{i=1}^{m-1}t(t_m-t_i)Q_i(u(t_i),u'(t_i))+(1-t)u_0+u_1t,\end{aligned}$$

于是对 $\forall t\in J$，$u,v\in B_r$，再结合条件(H_3)和引理 1，可得

$$\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)\leqslant\varphi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s(s-\tau)^{(\beta-1)/(1-\sigma)}\mathrm{d}\tau\right)^{1-\sigma}\left(\int_0^s\mu(\tau)^{1/\sigma}\mathrm{d}\tau\right)^{\sigma}\leqslant M\|\mu\|_{L^{1/\sigma}}^{q-1},$$

从而

$$\|Pu+Qv\|\leqslant M\|\mu\|_{L^{\lambda/\sigma}}^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)+2mL_I+(4m-2)L_Q+|u_0|+|u_1|.$$

因此 $Pu+Qv\in B_r$，由定理 2 和定理 3 知，边值问题(1)至少存在一个解.

定理 5 假设 $1<p<2$，若条件$(H_1),(H_2)$成立，且满足

$$\rho=N(q-1)\left(\frac{1}{\Gamma(\beta+1)}\right)^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)+2mL_I+(4m-2)(L_2+L_3)<1,\tag{6}$$

则边值问题(1)有唯一解.

证明：设 B_r 为 PC'的有界子集，令 $B_r=\{u\in PC'\mid\|u\|\leqslant r\}$，由式(6)可选取 r 为如下形式：

$$r>N\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{5m-1}{\Gamma(\alpha)}\right)+3mL_I+(7m-3)L_Q+3|u_0|+2|u_1|.$$

定义算子 $T:B_r\to PC'$，由定理 3 可知 Tu 一致有界，即有 $|(Tu)(t)|+|(Tu)'(t)|\leqslant L_1+L_2<r$，因此得 $\|(Tu)\|<r$，即 $T(B_r)\subseteq B_r$，往证 T 是压缩算子.

对于 $\forall t\in J$，$u,v\in B_r$，有

$$
\begin{aligned}
|(Tu)(t)-(Tv)(t)|\leqslant{}&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\\
&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\\
&\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\\
&\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\sum_{i=1}^{k}|I_i(u)-I_i(v)|+\\
&\sum_{i=1}^{k}(t-t_i)|Q_i(u,u')-Q_i(v,v')|+\sum_{i=1}^{k-1}(t_k-t_i)|Q_i(u,u')-Q_i(v,v')|+\\
&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\\
&\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\\
&\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,u,u')\mathrm{d}\tau\right)-\right.\\
&\left.\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\lambda f(\tau,v,v')\mathrm{d}\tau\right)\right|\mathrm{d}s+\sum_{i=1}^{k}|I_i(u)-I_i(v)|+\\
&\sum_{i=1}^{m}|I_i(u)-I_i(v)|+\sum_{i=1}^{m}(1-t_m)|Q_i(u,u')-Q_i(v,v')|+\\
&\sum_{i=1}^{m-1}(t_m-t_i)|Q_i(u,u')-Q_i(v,v')|+\sum_{i=1}^{m}|Q_i(u,u')-Q_i(v,v')|\leqslant\\
&\left\{N(q-1)\left(\frac{\lambda}{\Gamma(\beta+1)}\right)^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)+2mL+(4m-2)(L_2+L_3)\right\}\|u-v\|.
\end{aligned}
$$

由式(6)可得 T 为压缩映射，由压缩映射原理知，具 p-Laplace 算子的分数阶脉冲微分方程奇异边值问题(1)在 PC' 有唯一解.

3　应用实例

考虑如下奇异边值问题：

$$
\begin{cases}
{}^{\mathrm{C}}D_{0+}^{1/2}\varphi_p({}^{\mathrm{C}}D_{0+}^{3/2}u(t))=t^{-1/3}\left(\dfrac{|u(t)|}{16}+\dfrac{|u'(t)|}{16}\right),\quad t\in(0,1),\\
\Delta u=\dfrac{|u(1/2)|}{20+|u(1/2)|},\quad \Delta u=\dfrac{|u(1/2)|}{20+|u(1/2)|}+\dfrac{|u'(1/2)|}{20+|u'(1/2)|},\\
u(0)=0,\quad u(1)=1,\quad {}^{\mathrm{C}}D_{0+}^{3/2}u(0)=0,
\end{cases}
\tag{7}
$$

其中 $\beta=\frac{1}{2}$，$\alpha=\frac{3}{2}$，$p=\frac{4}{3}$，$\lambda=m=1$，$f(t,u,u')=t^{-1/3}\left(\frac{|u(t)|}{16}+\frac{|u'(t)|}{16}\right)$，则有

$$\rho=N(q-1)\left(\frac{1}{\Gamma(\beta+1)}\right)^{q-1}\left(\frac{2m+2}{\Gamma(\alpha+1)}+\frac{4m-2}{\Gamma(\alpha)}\right)+2mL_1+(4m-2)(L_2+L_3)=$$

$$\frac{224}{512\pi^2}+\frac{11}{20}\approx 0.594\ 3<1,$$

由定理5知，边值问题(7)有唯一解.

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