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一类具有多个吸引子的超混沌系统复杂动力学研究

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摘 要：针对高维Lorenz型系统复杂动力学研究不足问题，本文基于三维Lorenz系统，通过设计线性反馈控制器，提出1个具有3条平衡点曲线的新四维超混沌系统．借助Lyapunov指数、分岔图与相图，对该新四维超混沌系统展开研究，发现其在一定参数条件下具有丰富的动力学性质：具有3条平衡点曲线；在存在唯一平衡点的情形下，发生混沌－环－环面－焦点演变过程；在存在平衡点曲线的情形下，基于不同参数，数值模拟发现系统存在多个稳定吸引子共存；具有倍周期分岔、反倍周期分岔、Hopf分岔等多种动力学行为．严格证明四维Hopf分岔的存在性，并利用数值方法验证．进一步设计混沌电路，获得该系统混沌、超混沌等复杂动力学行为的物理验证，所得结果验证理论及数值模拟结果的有效性．

关 键 词：四维超混沌系统；平衡点曲线；共存吸引子；Hopf分岔；电路实现
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Study on Complex Dynamics of a Hyperchaotic System with Multiple Attractors

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Abstract：To address the insufficient research on the complex dynamics of high-dimensional Lorenz-type systems，a new four-dimensional hyperchaotic system with three equilibrium point curves was proposed by designing a linear feedback controller based on the three-dimensional Lorenz system. The dynamical properties of this new system were investigated by using Lyapunov exponents，bifurcation diagrams，and phase portraits. It is found that under certain parameter conditions，the system exhibits rich dynamical behaviors：it possesses three equilibrium point curves；under the condition of a unique equilibrium point，an evolution process from chaos to limit cycle，torus，and focus is observed；in the presence of equilibrium point curves，the coexistence of multiple stable attractors under different parameters is demonstrated by numerical simulations；various dynamical phenomena including period-doubling bifurcation，inverse period-doubling bifurcation，and Hopf bifurcation are exhibited. The existence of the four-dimensional Hopf bifurcation was strictly proven and verified numerically. Furthermore，a chaotic circuit was designed to physically verify the complex dynamical phenomena of the system，such as chaos and hyperchaos. The obtained results verify the validity of theoretical and numerical simulation results.

Key words：four-dimensional hyperchaotic system；equilibrium point curve；coexisting attractor；Hopf bifurcation；circuit implementation

自Lorenz在1963年发现Lorenz系统以来[1]，在过去几十年中，相关研究得到迅猛发展，混沌

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系统和超混沌系统不断涌现[2-6].Smale等一批数学家从理论层面展开探索,随后Rössler系统、Chua系统等相继被提出.此后,通过反馈控制技术构建出Lorenz系统的对偶形式Chen系统[7];Lyu系统揭示Lorenz系统与Chen系统之间的过渡关系[8];基于规范性理论,Yang系统通过分析2个稳定平衡点和1个不稳定平衡点的特征,明确同宿轨、异宿轨与混沌系统之间的内在关联[4,9],同时许多三维混沌系统被研究.

1979年Rössler[10]在三维Rössler系统的基础上提出1个四维超混沌系统,从而开启了对超混沌系统的研究[11-12],并发现超混沌系统在气象研究、图像加密、神经网络、无人驾驶等领域有极大的研究价值[13-18].Sukharev等[13]运用分岔分析等方法,拓展了Lorenz型吸引子的研究边界,为大气对流模拟及高维非线性系统混沌控制提供一定依据.Reyes等[15]运用递归图分析、Lyapunov指数等方法,完善了高维超混沌系统动力学分析与同步控制体系,为保密通信及跨领域高维非线性系统提供可迁移的分析与控制策略.Fan等[18]使用忆阻器构建偏置控制网格共存超混沌系统,分析所构建系统的不稳定不变集及共存超混沌吸引子,设计多通道循环数据加密方案,提高分布式安全传输性能.相较于三维混沌系统,四维超混沌系统具有更加复杂且丰富的动力学特征,可出现平衡点曲线、共存吸引子、具有2个正的Lyapunov指数的超混沌等复杂动力学行为,并蕴含高维Hopf分岔、倍周期与反倍周期分岔等复杂动力学行为.但是,关于四维超混沌系统的研究尚不充分:平衡点曲线、共存吸引子、环面及高维分岔等复杂动力学行为及现象缺乏探讨,平衡点曲线与其他动力学行为之间的内在关联仍不明确,高维分岔的理论分析及其工程应用也有待深入.

本文在经典的Lorenz系统中通过设计1个线性反馈控制器,在特定参数条件下获得1个具有1个、2个孤立平衡点、3条平衡点曲线、共存吸引子、环面、四维Hopf分岔等动力学行为的四维超混沌系统,这些有趣现象的存在可由分岔图、Lyapunov指数计算及电路仿真模拟验证.

1 新四维Lorenz型超混沌系统

本章基于经典的三维Lorenz系统,发现1个新型四维超混沌系统,该系统具有1个、2个孤立平衡点、3条平衡点曲线,并蕴含共存吸引子、Hopf分岔等特殊动力学行为.

1963年Lorenz提出经典的三维Lorenz系统

$$\left.\begin{aligned}\dot{x}&=a(y-x),\\ \dot{y}&=x(c-z)-y,\\ \dot{z}&=xy-bz.\end{aligned}\right\} \tag{1}$$

当系统参数$a=10,b=8/3,c=28$时,系统(1)发生混沌现象,并产生“蝴蝶效应”.

一般而言,产生超混沌的系统需满足必要条件:①系统方程的维数至少是4;②系统具有2个正的Lyapunov指数.因此,针对Lorenz系统第二、第三个方程,可设计1个相同的线性反馈控制器w,得到四维新系统:

$$\left.\begin{aligned}\dot{x}&=a(y-x),\\ \dot{y}&=cx-my-xz+w,\\ \dot{z}&=xy-bz-w,\\ \dot{w}&=k_1x+y-k_2z.\end{aligned}\right\} \tag{2}$$

其中系统参数$a>0,b>0,k_1\leqslant 100,k_2\leqslant 0,c,m$为实常数.若在系统(2)选取参数$a=15,b=5,c=-25,m=1,k_1=15,k_2=-19.3$,初值为(0,0,0,1 000),则系统(2)产生超混沌现象,在此参数下可得Lyapunov指数为$\lambda_1=1.012\,7,\lambda_2=0.231\,4,\lambda_3=0.000\,0,\lambda_4=-22.244\,2$,计算得其Lyapunov维数为3.056,故系统(2)是四维超混沌系统.其相图及Lyapunov指数如图1所示.

若选$a=10,b=8/3,c=28,m=1,k_1=-91.6,k_2=-2$,初值为(1,1,1,1),则系统(2)产生混沌现象,此参数下可得Lyapunov指数为$\lambda_1=0.161\,4,\lambda_2=0.000\,0,\lambda_3=-0.986\,9,\lambda_4=-12.841\,2$,计算得其Lyapunov维数为2.164,故系统(2)是四维混沌系统.其相图及Lyapunov指数如图2所示.

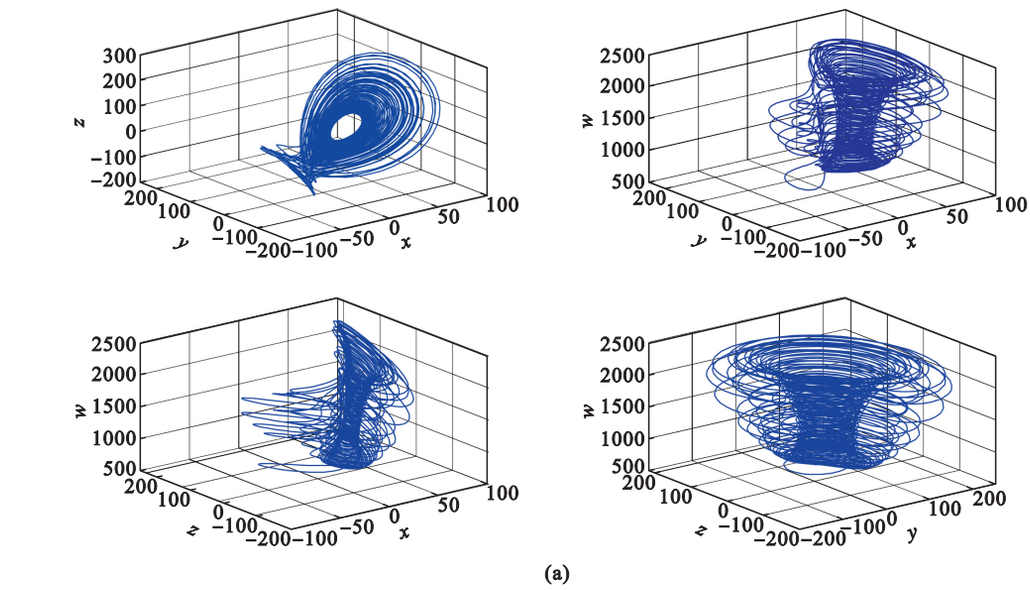
2 系统的平衡点、平衡点曲线及局部稳定性

系统(2)的散度为

$$\nabla\cdot V=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}+\frac{\partial\dot{w}}{\partial w}=-a-m-b. \tag{3}$$

因而,当$a+m+b>0$时,系统(2)是耗散的,于是当初始体积为V_0时,其体积按照公式$V=V_0\mathrm{e}^{-(a+m+b)t}$随时间耗散.由于$a+m+b$与系统变量$x,y,z,w$无关,故参数固定时,随着$t\to\infty$,系

统(2)以指数级的速度耗散至趋于0的区域.当 $a+m+b=0$ 时,这个四维超混沌系统(2)是保守的.



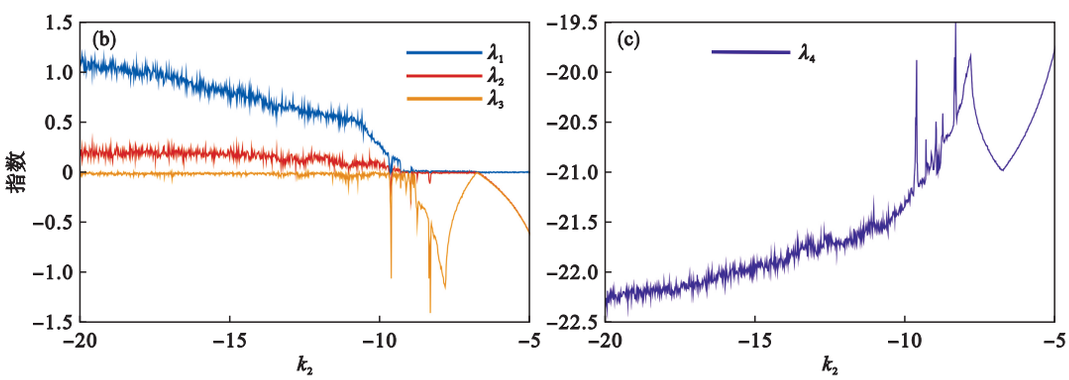
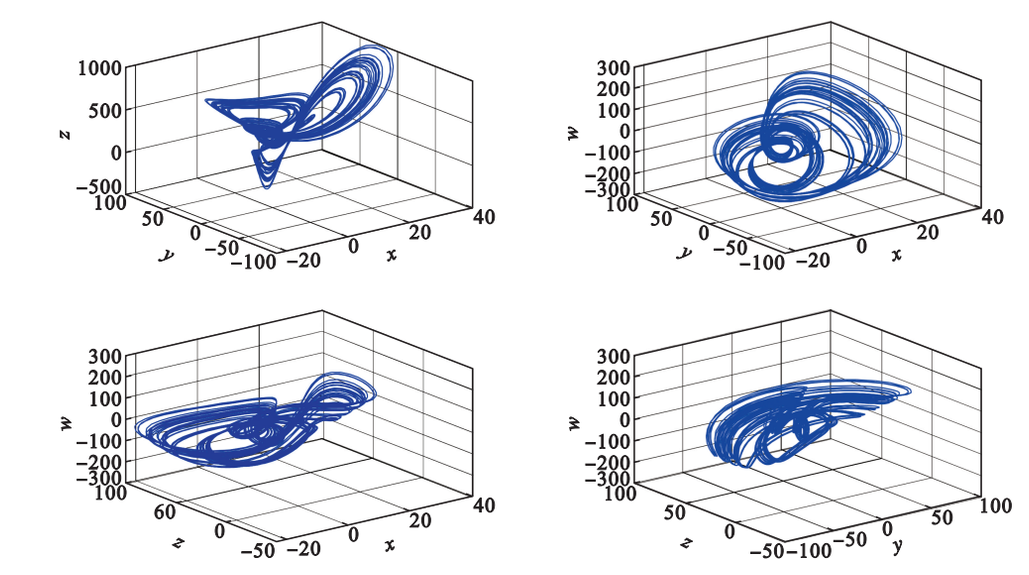


图1　系统(2)的动力学行为,参数 $(a,b,c,m,k_1,k_2)=(15,5,-25,1,15,-19.3)$

Fig. 1　Dynamical behaviors of system (2), parameters $(a,b,c,m,k_1,k_2)=(15,5,-25,1,15,-19.3)$

(a)—超混沌吸引子;(b),(c)—Lyapunov指数.



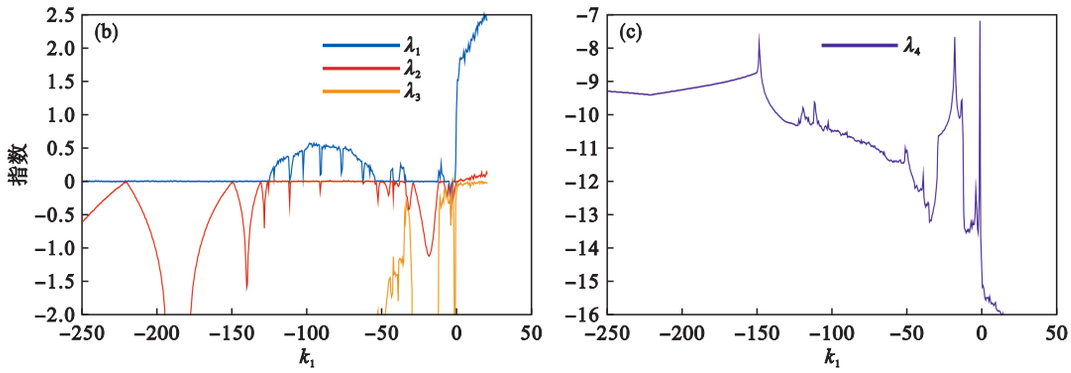


图2 系统(2)的动力学行为，参数$(a,b,c,m,k_1,k_2)=(10,8/3,28,1,-91.6,-2)$

Fig. 2 Dynamical behaviors of system (2),parameters $(a,b,c,m,k_1,k_2)=(10,8/3,28,1,-91.6,-2)$

(a)—混沌吸引子；(b),(c)—Lyapunov指数.

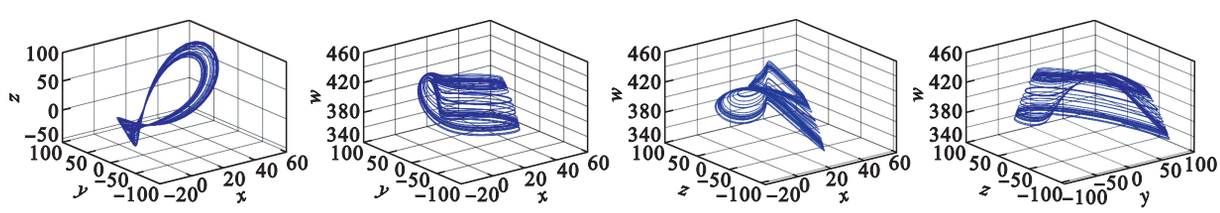
考虑系统(2)的平衡点个数，在$k_1-k_2+1\neq 0$的情形下，若$(k_1+1)b-(c-m)k_2=0$，则系统(2)只有1个平衡点；若$(k_1+1)b-(c-m)k_2\neq 0$，则存在除原点$O(0,\ 0,\ 0,\ 0)$外的另1个平衡点O'，其坐标为

$$x=\frac{k_2(c-m)-b(k_1+1)}{k_1-k_2+1},y=\frac{k_2(c-m)-b(k_1+1)}{k_1-k_2+1},$$

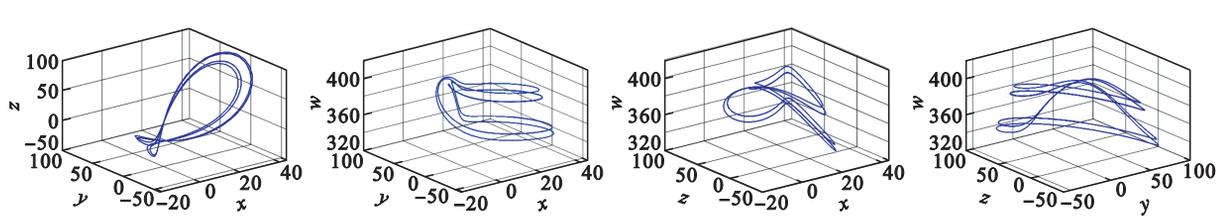
$$z=\frac{-(k_1+1)(b(1+k_1)-k_2(c-m))}{k_2(k_1-k_2+1)},$$

$$w=\frac{(b(1+k_1)-k_2(c-m))(b(k_1+1)^2-k_2^2(c-m))}{k_2(k_1-k_2+1)^2}.$$

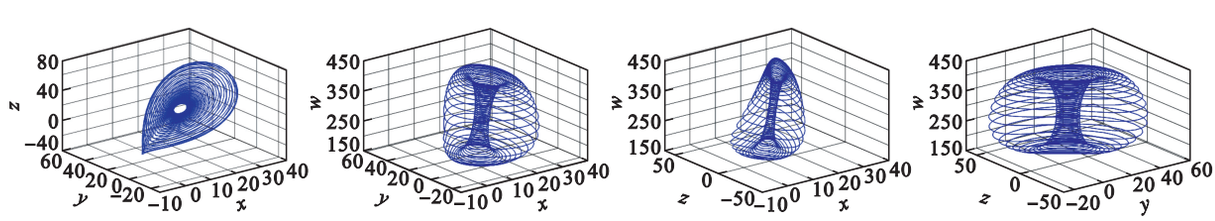
在$k_1-k_2+1=0$的情形下，若系统(2)参数满足$k_2(c-m-b)=0$，则系统(2)出现平衡点曲线；若参数$k_2(c-m-b)\neq 0$时，则系统(2)只有1个平衡点.选取系统(2)参数$(a,c,m,k_1,k_2)=(10,5,1,-16,-15)$，参数$b\in[0,4]$，系统(2)发生混沌-环-环面-焦点演变；取初值$(1,1,1,1)$，$b\in[0,0.13]$时，系统(2)出现混沌；当$b\in[0.13,0.40)$时，发生反倍周期分岔；当$b\in[0.40,3.99]$时，出现极限环及环面；当$b=4$时，收敛到稳定焦点，如图3所示.环面是高维系统中的1种特殊动力学行为，环面发生时系统的第一、第二Lyapunov指数为0.当$b=2.164$时系统(2)产生环面，其Lyapunov指数为$\lambda_1=0.000,\lambda_2=0.000,\lambda_3=-0.088,\lambda_4=-13.076$.



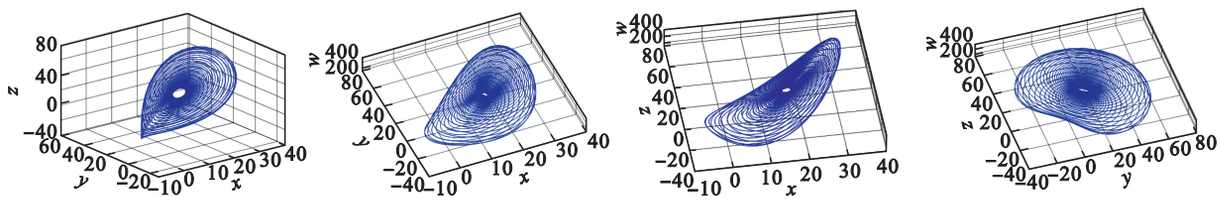
(a)



(b)



(c)



(d)

图3　不同b值下系统(2)的相图

Fig. 3　Phase diagrams of system (2) under different values of b

(a)—b=0.1;(b)—b=0.15;(c)—b=2.164;(d)—b=2.164(旋转相图).

对于单参数化自治动力系统$\dot{x}=f(x,\mu)$(μ为系统参数),平衡点曲线是参数-状态乘积空间$\mathbf{R}\times\mathbf{R}^n$中满足$f(x^*,\mu)=0$的点集的1个一维连通光滑不变流形.在平衡点曲线上,系统状态变量的一阶导数为0,各状态不随时间演化.物理意义下,平衡点曲线上点的变化速率为0,处于静止态.考虑平衡点曲线,当$k_1=-1,k_2=0,c-m-b\neq 0$时,系统(2)有2条平衡点曲线,记为

$$\Gamma_1\left(x,x,\frac{x^2+(c-m)x}{x+b},\frac{x^3-b(c-m)x}{x+b}\right),\quad x\in\mathbf{R}\setminus\{-b\}. \tag{4}$$

在Γ_1的参数取值下,$x=-b$处没有平衡点,系统(2)发生退化.选取初值为(x_0,y_0,z_0,w_0),若参数$c=m$,则系统(2)退化为

$$\left.\begin{aligned}&\dot{x}=a(y-x),\\&\dot{y}=\left(c+\frac{1}{a}\right)x-my-xz-\frac{x_0}{a}+w_0,\\&\dot{z}=xy-bz-\frac{1}{a}x+\frac{x_0}{a}-w_0.\end{aligned}\right\} \tag{5}$$

系统(2)分布于四维超平面$\Lambda_k=\{(x,y,z,w)|(x,y,z,w)\in\mathbf{R}^4,x-aw=x_0-aw_0\}$.在系统(5)中,系统(2)的平衡点曲线转化为平衡点,平衡点中的x分量满足$ax^3-x^2+(x_0-aw_0-b)x+bx_0-abw_0=0$.利用卡尔达诺公式,令$u=\frac{1}{9a^2}+\frac{b-x_0+aw_0}{3a}$,$v=\frac{1}{27a^3}-\frac{bx_0-abw_0}{2a}+\frac{b-x_0+aw_0}{6a^2}$,$n=\sqrt[3]{\sqrt{v^2-u^3}+v}$.当$a\neq 0$或$3a(b-x_0+aw_0)+1\neq 0$时,系统(5)的3个平衡点坐标中的$x$分量分别为

$$x_1=n+\frac{1}{3a}+\frac{u}{n},x_2=\frac{1}{3a}-\frac{\sqrt{3}\left(n-\frac{u}{n}\right)\mathrm{i}}{2}-\frac{n}{2}-\frac{u}{2n},x_3=\frac{1}{3a}+\frac{\sqrt{3}\left(n-\frac{u}{n}\right)\mathrm{i}}{2}-\frac{n}{2}-\frac{u}{2n}.$$

当$x_0-aw_0=0$时,系统(5)中3个平衡点坐标的x分量为$x_1=0,x_2=-\frac{\sqrt{4ab+1}-1}{2a},x_3=\frac{\sqrt{4ab+1}+1}{2a}$.

Γ_1上的平衡点处Jacobi矩阵的特征方程为

$$f(\lambda)=\lambda^4+(a+b+m)\lambda^3+\left(am+ab-ac+mb-1+x^2+\frac{ax^2-amx-abc}{x+b}\right)\lambda^2+\left(abm-abc-b+2ax^2-x+ab\frac{x^2-mx-bc}{x+b}\right)\lambda,$$

分别记特征方程$f(\lambda)$的各次系数为$p_0^{\Gamma_1}=abm-abc-b+2ax^2-x+ab\frac{x^2-mx-bc}{x+b}$,$p_1^{\Gamma_1}=am+ab-ac+mb-1+x^2+\frac{ax^2-amx-abc}{x+b}$,$p_2^{\Gamma_1}=a+b+m$.

根据Hurwitz判据可知,系统(2)在Γ_1上平衡点处具有1个中心流形,3个稳定流形的必要条件为

$$p_0^{\Gamma_1}>0,\ p_1^{\Gamma_1}>0,\ p_2^{\Gamma_1}>0,\ p_1^{\Gamma_1}p_2^{\Gamma_1}-p_0^{\Gamma_1}>0. \tag{6}$$

当$k_1-k_2+1=0$且$c-m-b=0$时,系统(2)有平衡点曲线:

$$\Gamma_2(x,x,x,x^2-bx),x\in\mathbf{R}. \tag{7}$$

Γ_2上的平衡点为奇异退化平衡点.

Γ_2上平衡点处Jacobi矩阵的特征方程为

$$f(\lambda)=\lambda^4+(a+c)\lambda^3+\left(-m^2+cm-k_1-2+x^2+ax\right)\lambda^2+\left(-ac^2+2acm-c-am^2-k_1m-2a-2ak_1+2ax^2+acx-amx+k_1x\right)\lambda,$$

分别记特征方程$f(\lambda)$的各次系数为$p_0^{\Gamma_2}=-ac^2+2acm-c-am^2-k_1m-2a-2ak_1+2ax^2+acx-amx+k_1x$,$p_1^{\Gamma_2}=-m^2+cm-k_1-2+x^2+ax$,$p_2^{\Gamma_2}=a+c$.

根据Hurwitz判据可知,系统(2)在Γ_2的平衡点处具有1个中心流形,3个稳定流形的必要条件为

$$p_0^{\Gamma_2}>0,\ p_1^{\Gamma_2}>0,\ p_2^{\Gamma_2}>0,\ p_1^{\Gamma_2}p_2^{\Gamma_2}-p_0^{\Gamma_2}>0. \tag{8}$$

对于参数化自治动力系统$\dot{x}=f(x,\mu)$,若在某

一固定参数值 $\mu=\mu_0$ 下，相空间 $\mathbf{R}^n$ 中存在 2 个或多个互不相交的吸引子 $A_1,A_2,\cdots,A_k(k\geqslant 2)$，则称系统在该参数下存在共存吸引子．由于 Γ_1,Γ_2 上有无穷多个平衡点，系统(2)出现多个吸引子共存现象．在 Γ_1 情形下，若选取参数 $(a,b,c,m,k_1,k_2)=(10,3,1,1,-1,0)$，并选取初值 $(r,|r|,|r|,|r|)$，其中 r 属于集合 $\{10^4,2\times10^4,3\times10^4,-10^4,-2\times10^4,-3\times10^4\}$，则发生多个稳定极限环共存现象，如图4a所示；在 Γ_2 情形下，若选取参数 $(a,b,c,m,k_1,k_2)=(10,3,4,1,-16,-15)$，并选取不同初值，则发生多个稳定焦点共存现象，如图4b，4c所示．

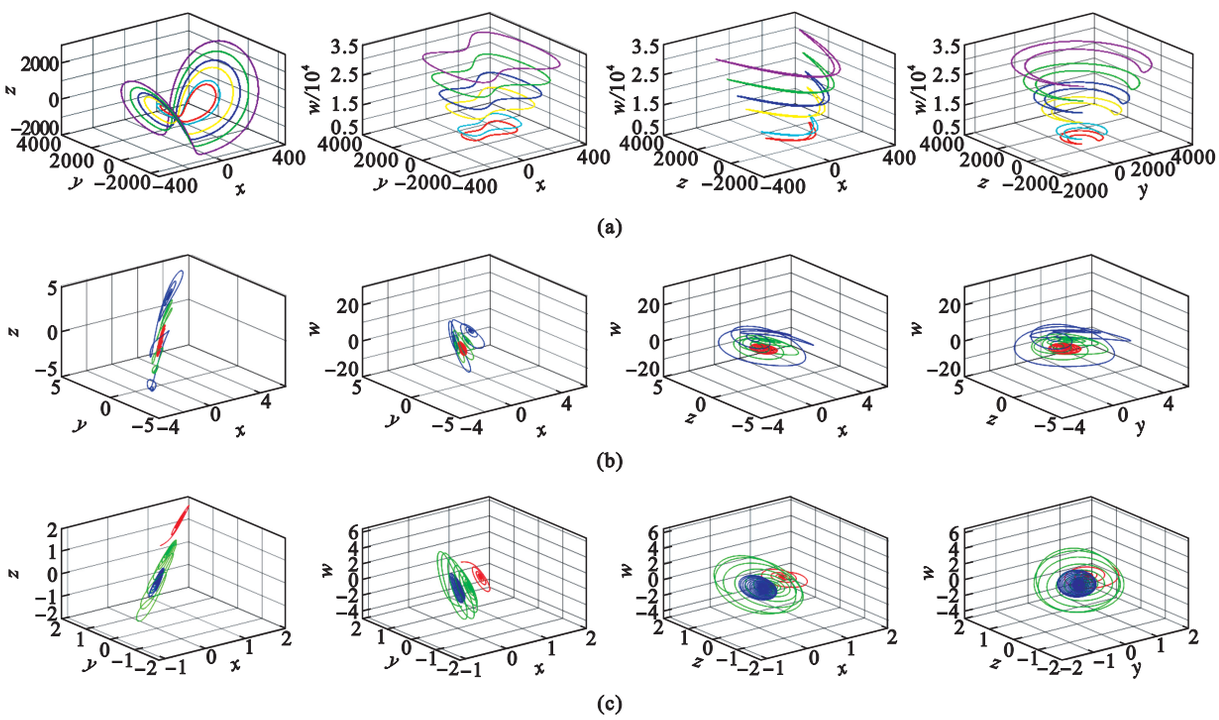
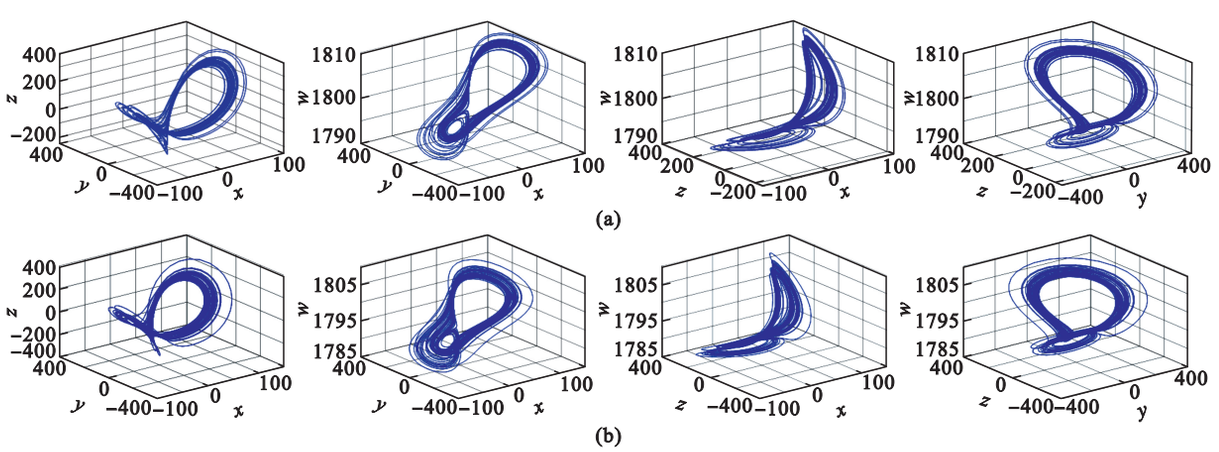


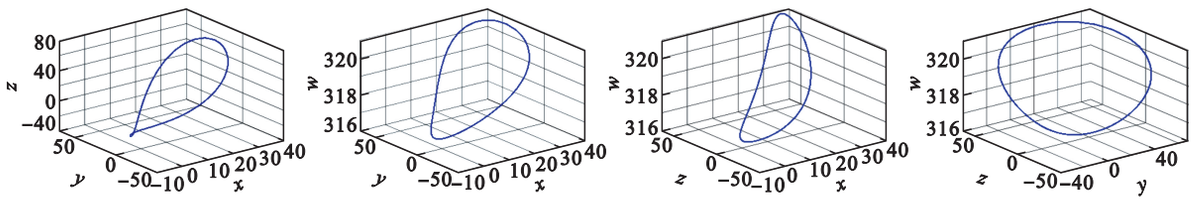
图4　共存吸引子

Fig. 4　Coexisting attractors

(a)—参数 $(a,b,c,m,k_1,k_2)=(10,3,1,1,-1,0)$，$\boldsymbol{r}\in\{10^4,2\times10^4,3\times10^4,-10^4,-2\times10^4,-3\times10^4\}$，分别对应红、黄、绿、蓝、深蓝、品红；(b)—参数 $(a,b,c,m,k_1,k_2)=(10,3,4,1,-16,-15)$，初值：红(1, 1.5, 0.5, 1)、绿(−2, −2, −2, 11)、蓝(−3, −3, −3, 19)；(c)—参数 $(a,b,c,m,k_1,k_2)=(10,3,4,1,-16,-15)$，初值：红(−1, −1, −1, 5)、绿(−0.5, −1, 0.2, −0.3)、蓝(0.1, 0.2, −0.5, 0.4).

对平衡点曲线的参数施加扰动致使其结构破裂，系统(2)中的多个吸引子共存现象消失，最终收敛于唯一稳定吸引子．若选取参数 $(a,b,c,m,k_1,k_2)=(10,3,1,1,-1,-0.001)$，则选取不同初值，系统(2)初期瞬态演化轨迹与参数 $(a,b,c,m,k_1,k_2)=(10,3,1,1,-1,0)$ 时的演化轨迹类似，最终收敛于唯一稳定极限环，如图5b，5c所示．在工程电路中，平衡点曲线的出现会使稳定周期振荡突变为混沌振荡，导致控制系统失稳．工程电路在参数设置时，应规避系统产生平衡点曲线，防止周期振荡消失．





(c)

图5　对Γ_1微小扰动下系统(2)的动力学行为

Fig. 5　Dynamical behaviors of system (2) under a small perturbation to Γ_1

(a)—Γ_1曲线存在:参数$(a, b, c, m, k_1, k_2)=(10, 3, 1, 1, -1, 0)$,初值(2 000,2 000,2 000,2 000);

(b)—Γ_1曲线消失:参数$(a, b, c, m, k_1, k_2)=(10, 3, 1, 1, -1, -0.001)$,初值同(a),暂态时间100;

(c)—Γ_1曲线消失:参数$(a, b, c, m, k_1, k_2)=(10, 3, 1, 1, -1, -0.001)$,初值同(a),暂态时间100 000.

下面讨论系统(2)在原点$O(0,0,0,0)$处的局部稳定性.系统(2)在原点处的Jacobi矩阵的特征方程为

$$f(\lambda)=\lambda^4+(a+b+m)\lambda^3+(ab-k_2-ac+am+bm-1)\lambda^2+$$
$$(abm-b-ak_1-ak_2-k_2m-abc-a)\lambda+ak_2(c-m)-ab(k_1+1),$$

分别记特征方程$f(\lambda)$的各次系数为$p_0=ak_2(c-m)-ab(k_1+1)$, $p_1=ab(m-c)-a(k_1+1)-k_2(a+m)-b_1$, $p_2=ab-k_2-ac+am+bm-1$, $p_3=a+b+m$, $b_1=\dfrac{p_2p_3-p_1}{p_3}$.

根据Hurwitz判据可知,系统(2)在原点O稳定的必要条件为

$$p_0>0,\ p_3p_2-p_1>0,\ b_1p_1-p_0p_3>0,\ p_3>0. \tag{9}$$

3　系统的动力学分析及数值模拟

3.1　Hopf分岔

定理1　若系统(2)的参数满足条件(9)且有条件

$$\left.\begin{aligned}
&K=\frac{ab(m-c)-a(k_1+k_2+1)-b-k_2m}{a+b+m}>0,\\
&p_1^2-p_1p_2p_3+p_3^2p_0=0,\\
&ack_2-abk_1-ab-ak_2m\neq 0,\\
&-(a+b+m)a(m-c)+(-a+b-m)K+(a+m)(ab+am+bm-ac-k_2-1)\neq 0.
\end{aligned}\right\} \tag{10}$$

成立,则系统(2)在原点O处发生Hopf分岔.

证明　系统(2)在原点$O(0,0,0,0)$处发生Hopf分岔需满足稳定性条件,将系统(2)原点O处的Jacobi矩阵的纯虚特征根$\mathrm{i}\omega$代入特征方程

$$f(\mathrm{i}\omega)=\omega^4-(ab-k_2-ac+am+bm-1)\omega^2+(ak_2(c-m)-ab(k_1+1))+$$
$$\mathrm{i}\omega\left[-(a+b+m)\omega^2+(ab(m-c)-a(k_1+1)-k_2(a+m)-b)\right],$$

因此由$f(\mathrm{i}\omega)=0$可得$\omega^2=\dfrac{ab(m-c)-a(k_1+k_2+1)-b-k_2m}{a+b+m}>0$, $p_1^2-p_1p_2p_3+p_3^2p_0=0$.

为使系统(2)的特征值不为0,要求$ak_2(c-m)-ab(k_1+1)\neq 0$.还需满足横截条件,利用隐函数定理:

$$F(\lambda,k_2)=\lambda^4+(a+b+m)\lambda^3+(ab-ac+am+bm-k_2-1)\lambda^2+$$
$$(abm-abc-a-b-ak_1-ak_2-k_2m)\lambda+(ack_2-abk_1-ab-ak_2m),$$

$$\frac{\partial F}{\partial\lambda}=4\lambda^3+3(a+b+m)\lambda^2+2(ab-ac+am+bm-k_2-1)\lambda+(abm-abc-a-b-ak_1-ak_2-k_2m),$$

$$\frac{\partial F}{\partial k_2}=-\lambda^2-(a+m)\lambda+a(c-m),$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}k_2}=\frac{\lambda^2+(a+m)\lambda-a(c-m)}{4\lambda^3+3(a+b+m)\lambda^2+2(ab-ac+am+bm-k_2-1)\lambda+(abm-abc-a-b-ak_1-ak_2-k_2m)},$$

$$\lambda(k_2)|_{\lambda=\mathrm{i}\omega}=\frac{\lambda^2+(a+m)\lambda-a(c-m)}{4\lambda^3+3(a+b+m)\lambda^2+2(ab-ac+am+bm-k_2-1)\lambda+(abm-abc-a-b-ak_1-ak_2-k_2m)}\Big|_{\lambda=\mathrm{i}\omega},$$

$$N(\mathrm{i}\omega)\triangleq\frac{\partial F}{\partial k_2}|_{\lambda=\mathrm{i}\omega}=a(m-c)-\omega^2+(a+m)\omega\mathrm{i},$$

$$D(\mathrm{i}\omega)\triangleq\frac{\partial F}{\partial\lambda}|_{\lambda=\mathrm{i}\omega}=-2(a+b+m)\omega^2+$$

$$2\omega\left(-2\omega^2+ab+am+bm-ac-k_2-1\right)\mathrm{i}.$$

$$\mathrm{Num}\triangleq\frac{\mathrm{Re}(N)\mathrm{Re}(D)+\mathrm{Im}(N)\mathrm{Im}(D)}{2\omega^2},|\partial P/\partial\lambda|^2=$$

$$\mathrm{Re}(D)^2+\mathrm{Im}(D)^2,\text{有 }\mathrm{Re}\left(\frac{N(\mathrm{i}\omega)}{D(\mathrm{i}\omega)}\right)=2\omega^2\frac{\mathrm{Num}}{|\partial P/\partial\lambda|^2},$$

$$\mathrm{Num}=-(a+b+m)a(m-c)+$$

$$\frac{[ab(m-c)-a(k_1+k_2+1)-b-k_2m](-a+b-m)}{a+b+m}+$$

$$(a+m)(ab+am+bm-ac-k_2-1)\neq0.$$

根据条件(9)和(10),可推得系统(2)在原点处发生Hopf分岔.

定理2 若条件(9)和(10),以及$(a,b,c,m)=(1,1,0,2)$成立,则系统(2)在原点处存在Hopf分岔,且有

(i) Hopf分岔是非退化、超临界且稳定的;

(ii) Hopf分岔的近似周期$T=\frac{2\pi}{\omega_0}(1+\tau_2\varepsilon^2+O(\varepsilon^4))$,$\beta=\beta_2\varepsilon^2+O(\varepsilon^4)$,其中$\beta$为极限环振幅的平方,

$$\tau_2=-\frac{-288k_1^8-2\,564k_1^7+25\,679k_1^6+393\,546k_1^5+1\,592\,165k_1^4+2\,318\,176k_1^3+672\,984k_1^2-277\,344k_1-33\,264}{18(k_1+2)^3(k_1+3)^2(k_1-6)^2(4k_1+11)(k_1-1)(4k_1+3)},$$

$$\beta_2=\frac{-96k_1^7-788k_1^6+8\,095k_1^5+122\,344k_1^4+535\,125k_1^3+986\,334k_1^2+730\,764k_1+182\,952}{3(k_1+2)^3(k_1-6)^2(k_1-1)(k_1+3)(4k_1+11)(4k_1+3)},$$

$$\omega_0=\sqrt{\frac{ab(m-c)-a(k_1+k_2+1)-b-k_2m}{a+b+m}},\varepsilon^2=\frac{k_2-k_2'}{\mu_2}+O((k_2-k_2')^2).$$

这里

$$\mu_2=\frac{-96k_1^7-788k_1^6+8\,095k_1^5+122\,344k_1^4+535\,125k_1^3+986\,334k_1^2+730\,764k_1+182\,952}{3(k_1+2)^3(k_1-6)(k_1-1)(k_1+3)(4k_1+11)(4k_1+3)}.$$

(iii) Hopf分岔周期解近似表达式为

$$x=\varepsilon\cos\left(\frac{2\pi t}{T}\right)-\varepsilon^2\left(\frac{6(k_1^2+5k_1-12)}{(k_1-1)(k_1+2)(k_1+3)(k_1-6)}\right)-$$

$$\varepsilon^2\left(\frac{\sin\left(\frac{4\pi t}{T}\right)(k_1^4+34k_1^3+139k_1^2+90k_1+96)}{3\sqrt{-k_1-3}\,(k_1-1)^2(k_1+2)^2(k_1-6)}+\right.$$

$$\left.\frac{\cos\left(\frac{4\pi t}{T}\right)(9k_1^4+98k_1^3+263k_1^2+182k_1+168)}{3(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right),$$

$$y=\varepsilon\sin\left(\frac{2\pi t}{T}\right)+\varepsilon^2\left(\frac{3k_1}{(k_1-1)(k_1+2)(k_1-6)\sqrt{-k_1-3}}\right)+$$

$$\varepsilon^2\left(\frac{\cos\left(\frac{4\pi t}{T}\right)(2k_1^4+41k_1^3+125k_1^2-180k_1-348)}{3(k_1-1)^2(k_1+2)^2(k_1-6)\sqrt{-k_1-3}}-\right.$$

$$\left.\frac{4\sin\left(\frac{4\pi t}{T}\right)(3k_1^4+28k_1^3+43k_1^2-104k_1-150)}{3(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right),$$

$$z=\frac{k_1\sqrt{\sigma_1{}^2+\sigma_2{}^2}}{(k_1-1)(k_1+2)(k_1-6)}+\frac{(\sigma_2{}^2-\sigma_1{}^2)(4k_1{}^4+29k_1{}^3+83k_1{}^2+160k_1+84)}{(4k_1+3)(k_1-6)(k_1+2)^2(k_1-1)^2}+\frac{24\sigma_1\sigma_2\sqrt{-k_1-3}\,(k_1{}^3+3k_1{}^2+7k_1+4)}{(4k_1+3)(k_1-6)(k_1+2)^2(k_1-1)^2},$$

$$w=-\frac{3k_1\sqrt{\sigma_1{}^2+\sigma_2{}^2}}{2(k_1-1)(k_1+2)^2}-\frac{k_1(\sigma_2{}^2-\sigma_1{}^2)(4k_1{}^3+43k_1{}^2+109k_1+60)}{2(4k_1+11)(k_1-1)^2(k_1+2)^3}-\frac{2k_1\sigma_1\sigma_2\sqrt{-k_1-3}\,(8k_1{}^2+29k_1+17)}{(4k_1+11)(k_1-1)^2(k_1+2)^3}.$$

其中:

$$\sigma_1=\varepsilon\cos\left(\frac{2\pi t}{T}\right)-\varepsilon^2\frac{(k_1{}^2+5k_1-12)}{(k_1-1)(k_1+2)(k_1+3)(k_1-6)}-$$

$$\varepsilon^2\left(\frac{\sin\left(\frac{4\pi t}{T}\right)(k_1{}^4+34k_1{}^3+139k_1{}^2+90k_1+96)}{3(k_1-1)^2(k_1+2)^2(k_1-6)\sqrt{-k_1-3}}+\frac{\cos\left(\frac{4\pi t}{T}\right)(9k_1^4+98k_1{}^3+263k_1{}^2+182k_1+168)}{3(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right);$$

$$\sigma_2=\varepsilon\sin\left(\frac{2\pi t}{T}\right)+\varepsilon^2\left(\frac{3k_1}{(k_1-1)(k_1+2)(k_1-6)\sqrt{-k_1-3}}\right)+$$

$$\varepsilon^2\left(\frac{\cos\left(\frac{4\pi t}{T}\right)(2k_1^4+41k_1{}^3+125k_1{}^2-180k_1-348)}{3(k_1-1)^2(k_1+2)^2(k_1-6)\sqrt{-k_1-3}}-\frac{4\sin\left(\frac{4\pi t}{T}\right)(3k_1^4+28k_1{}^3+43k_1{}^2-104k_1-150)}{(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right).$$

证明　选取满足条件(9),(10)的参数(a,b,c,m,k_1,k_2),并取$t_1=\sqrt{\dfrac{ab(m-c)-a(k_1+k_2+1)-b-k_2m}{a+b+m}}$,记$k_2$为$k_2'$.再取$(a,b,c,m,k_1,k_2)=(1,1,0,2,k_1,k_2)$,于是,$t_1=\sqrt{-(k_1+3)}$,$k_2'=k_1+4$而知特征方程

$$f(\lambda)=(\lambda+1)(\lambda+3)(\lambda^2-(k_1+3)).$$

当$k_1+3<0$时,则有特征值及对应的特征向量分别为

$$\lambda_{e1}=\mathrm{i}\sqrt{-(k_1+3)},\lambda_{e2}=-\mathrm{i}\sqrt{-(k_1+3)},\lambda_{e3}=-3,\lambda_{e4}=-1,$$

$$\boldsymbol{v}_1=\begin{pmatrix}\dfrac{k_1+5}{(k_1-1)(k_1+2)}-\mathrm{i}\dfrac{3\sqrt{-(k_1+3)}}{(k_1-1)(k_1+2)}\\ -\dfrac{2}{k_1-1}+\mathrm{i}\dfrac{\sqrt{-(k_1+3)}}{k_1-1}\\ \dfrac{1}{k_1+2}-\mathrm{i}\dfrac{\sqrt{-(k_1+3)}}{k_1+2}\\ 1\end{pmatrix},\boldsymbol{v}_3=\begin{pmatrix}\dfrac{1}{2}\\ -1\\ \dfrac{1}{2}\\ 1\end{pmatrix},\boldsymbol{v}_4=\begin{pmatrix}\dfrac{k_1+4}{k_1}\\ 0\\ 1\\ 0\end{pmatrix},$$

$$\boldsymbol{P}=\left(\mathrm{Re}(\boldsymbol{v}_1),\mathrm{Im}(\boldsymbol{v}_1),\boldsymbol{v}_3,\boldsymbol{v}_4\right)=\begin{pmatrix}\dfrac{k_1+5}{k_1^2+k_1-2} & -\dfrac{3\sqrt{-(k_1+3)}}{k_1^2+k_1-2} & \dfrac{1}{2} & \dfrac{k_1+4}{k_1}\\ -\dfrac{2}{k_1-1} & \dfrac{\sqrt{-(k_1+3)}}{k_1-1} & -1 & 0\\ \dfrac{1}{k_1+2} & -\dfrac{\sqrt{-(k_1+3)}}{k_1+2} & \dfrac{1}{2} & 1\\ 1 & 0 & 1 & 0\end{pmatrix}.$$

$(x_1,y_1,z_1,w_1)^{\mathrm{T}}=\boldsymbol{P}(x_2,y_2,z_2,w_2)^{\mathrm{T}}$.　　系统(2)转化为

$$\dot{x}_2=\sqrt{-k_1-3}\,y_2+F_1(x_2,y_2,z_2.w_2),\dot{y}_2=-\sqrt{-k_1-3}\,x_2+F_2(x_2,y_2,z_2,w_2),$$
$$\dot{z}_2=-3z+F_3(x_2,y_2,z_2,w_2),\dot{w}_2=-w_2+F_4(x_2,y_2,z_2,w_2).$$

其中:

$$F_1(x_2,y_2,z_2,w_2)=-\frac{(2k_1^2+4k_1-16)w_2^2}{k_1(k_1-6)}+\frac{6\left(-k_1^3-8k_1^2-21k_1-30\right)x_2^2}{(k_1-6)(k_1^2+k_1-2)^2}+\frac{6(2k_1^3+9k_1^2+19k_1+30)y^2}{(k_1-6)(k_1^2+k_1-2)^2}-$$
$$\frac{(3k_1+6)z^2}{2(k_1-6)}-\frac{8(k_1^3+6k_1^2+11k_1+18)x_2w_2}{k_1(k_1-6)(k_1+2)(k_1-1)}-\frac{(4k_1^2+16k_1+24)z_2w_2}{k_1(k_1-6)}-\frac{6(k_1^2+5k_1+8)x_2z_2}{(k_1-6)(k_1+2)(k_1-1)}+$$
$$\frac{\sqrt{-k_1-3}\cdot4(k_1^3+7k_1^2+8k_1+20)y_2w_2}{k_1(k_1-6)(k_1+2)(k_1-1)}+\frac{\sqrt{-k_1-3}\cdot4\left(k_1^4+11k_1^3+26k_1^2+52k_1\right)x_2y_2}{k_1(k_1-6)(k_1^2+k_1-2)^2}+$$
$$\frac{\sqrt{-k_1-3}\cdot2(k_1^2+6k_1+14)y_2z_2}{(k_1-6)(k_1-1)(k_1+2)};\ F_2(x_2,y_2,z_2,w_2)=\frac{4\left(k_1+4\right)w_2^2}{\sqrt{-k_1-3}}+\frac{4\left(k_1+5\right)(k_1^3+4k_1^2-11k_1-24)x_2^2}{\sqrt{-k_1-3}\,(k_1-6)(k_1+2)^2(k_1-1)^2}-$$
$$\frac{6\left(k_1+3\right)(k_1^3+5k_1^2-12k_1-24)y_2^2}{\sqrt{-k_1-3}\,(k_1-6)(k_1+2)^2(k_1-1)^2}+\frac{\left(k_1^2+2k_1-12\right)z_2^2}{\sqrt{-k_1-3}\,(k_1-6)}+\frac{4\left(k_1^2+2k_1-12\right)(k_1^2+7k_1+8)w_2x_2}{k_1\sqrt{-k_1-3}\,(k_1-6)(k_1+2)(k_1-1)}+$$
$$\frac{2\left(k_1+3\right)(k_1^2+4k_1-16)w_2z_2}{k_1\sqrt{-k_1-3}\,(k_1-6)}+\frac{2\left(k_1+3\right)(2k_1^2+5k_1-28)x_2z_2}{\sqrt{-k_1-3}\,(k_1-6)(k_1+2)(k_1-1)}-\frac{2\left(k_1^4+9k_1^3+14k_1^2-72k_1-96\right)w_2y_2}{k_1(k_1-6)(k_1+2)(k_1-1)}-$$
$$\frac{2\left(k_1^4+16k_1^3+37k_1^2-150k_1-264\right)x_2y_2}{(k_1-6)(k_1+2)^2(k_1-1)^2}-\frac{\left(k_1^3+11k_1^2-96\right)y_2z_2}{(k_1-6)(k_1+2)(k_1-1)};\ F_3(x_2,y_2,z_2,w_2)=\frac{2\left(k_1+4\right)(k_1-2)w_2^2}{k_1(k_1-6)}+$$
$$\frac{6\left(k_1^2+3k_1+6\right)(k_1+5)x_2^2}{(k_1-6)(k_1+2)^2(k_1-1)^2}-\frac{6\left(k_1+3\right)(2k_1^2+3k_1+10)y_2^2}{(k_1-6)(k_1+2)^2(k_1-1)^2}+\frac{3\left(k_1+2\right)z_2^2}{2(k_1-6)}+\frac{8\left(k_1^3+6k_1^2+11k_1+18\right)w_2x_2}{k_1(k_1-6)(k_1+2)(k_1-1)}+$$
$$\frac{4\left(k_1^2+4k_1+6\right)w_2z_2}{k_1(k_1-6)}+\frac{6\left(k_1^2+5k_1+8\right)x_2z_2}{2(k_1-6)(k_1+2)(k_1-1)}-\frac{4\sqrt{-k_1-3}\left(k_1^3+7k_1^2+8k_1+20\right)w_2y_2}{k_1(k_1-6)(k_1+2)(k_1-1)}-$$

$$\frac{4\sqrt{-k_1-3}\left(k_1^3+11k_1^2+26k_1+52\right)x_2y_2}{(k_1-6)(k_1+2)^2(k_1-1)^2}-\frac{2\sqrt{-k_1-3}\left(k_1^2+6k_1+14\right)y_2z_2}{(k_1-6)(k_1+2)(k_1-1)};\ F_4(x_2,y_2,z_2,w_2)=$$

$$-\frac{\left(k_1+4\right)w_2^2}{k_1+2}-\frac{3k_1\left(k_1+5\right)(k_1+1)x_2^2}{(k_1-1)^2(k_1+2)^3}+\frac{3k_1\left(k_1+3\right)(2k_1+1)y_2^2}{(k_1-1)^2(k_1+2)^3}-\frac{3k_1z_2^2}{4(k_1+2)}-\frac{4\left(k_1^2+5k_1+3\right)w_2x_2}{(k_1-1)(k_1+2)^2}-$$

$$\frac{2\left(k_1+3\right)w_2z_2}{(k_1+2)}-\frac{3k_1\left(k_1+3\right)x_2z_2}{(k_1-1)(k_1+2)^2}+\frac{2\sqrt{-k_1-3}\left(k_1^2+6k_1+2\right)w_2y_2}{(k_1-1)(k_1+2)^2}+\frac{2\sqrt{-k_1-3}\,k_1\left(k_1^2+10k_1+7\right)x_2y_2}{(k_1-1)^2(k_1+2)^3}+$$

$$\frac{\sqrt{-k_1-3}\,k_1\left(k_1+5\right)y_2z_2}{(k_1-1)(k_1+2)^2}.$$

进一步计算可得

$$g_{11}=\frac{1}{4}\left[\frac{\partial^2F_1}{\partial x_2^2}+\frac{\partial^2F_1}{\partial y_2^2}+\left(\frac{\partial^2F_2}{\partial x_2^2}+\frac{\partial^2F_2}{\partial y_2^2}\right)\mathrm{i}\right]=\frac{3k_1}{(k_1-1)(k_1+2)(k_1-6)}+\left(\frac{k_1^2+5k_1-12}{-\sqrt{-k_1-3}\,(k_1-1)(k_1+2)(k_1-6)}\right)\mathrm{i},$$

$$g_{02}=\frac{1}{4}\left[\frac{\partial^2F_1}{\partial x_2^2}-\frac{\partial^2F_1}{\partial y_2^2}-2\frac{\partial^2F_2}{\partial x_2\partial y_2}+\left(\frac{\partial^2F_2}{\partial x_2^2}-\frac{\partial^2F_2}{\partial y_2^2}+2\frac{\partial^2F_1}{\partial x_2\partial y_2}\right)\mathrm{i}\right]=$$

$$\frac{k_1^3+5k_1^2-24k_1-222}{(k_1-1)^2(k_1+2)(k_1-6)}+\left(\frac{3k_1^3+8k_1^2-107k_1-384}{\sqrt{-k_1-3}\,(k_1-1)^2(k_1+2)(k_1-6)}\right)\mathrm{i},$$

$$g_{20}=\frac{1}{4}\left[\frac{\partial^2F_1}{\partial x_2^2}-\frac{\partial^2F_1}{\partial y_2^2}+2\frac{\partial^2F_2}{\partial x_2\partial y_2}+\left(\frac{\partial^2F_2}{\partial x_2^2}-\frac{\partial^2F_2}{\partial y_2^2}-2\frac{\partial^2F_1}{\partial x_2\partial y_2}\right)\mathrm{i}\right]=-\frac{k_1^3+26k_1^2+114k_1+84}{(k_1-1)(k_1+2)^2(k_1-6)}+$$

$$\left(\frac{7k_1^3+77k_1^2+222k_1+144}{\sqrt{-k_1-3}\,(k_1-1)(k_1+2)^2(k_1-6)}\right)\mathrm{i},$$

$$G_{21}=\frac{1}{8}\left[\frac{\partial^3F_1}{\partial x_2^3}+\frac{\partial^3F_1}{\partial x_2\partial y_2^2}+\frac{\partial^3F_2}{\partial x_2^2\partial y_2}+\frac{\partial^3F_2}{\partial y_2^3}+\left(\frac{\partial^3F_2}{\partial x_2^3}+\frac{\partial^3F_2}{\partial x_2\partial y_2^2}-\frac{\partial^3F_1}{\partial x_2^2\partial y_2}-\frac{\partial^3F_1}{\partial y_2^3}\right)\mathrm{i}\right]=0,$$

$$h_{11}^1=\frac{1}{4}\left(\frac{\partial^2F_3}{\partial x_2^2}+\frac{\partial^2F_3}{\partial y_2^2}\right)=-\frac{3k_1}{(k_1-1)(k_1+2)(k_1-6)},h_{11}^2=\frac{1}{4}\left(\frac{\partial^2F_4}{\partial x_2^2}+\frac{\partial^2F_4}{\partial y_2^2}\right)=\frac{3k_1}{2(k_1-1)(k_2+2)^2},$$

$$h_{20}^1=\frac{1}{4}\left(\frac{\partial^2F_3}{\partial x_2^2}-\frac{\partial^2F_3}{\partial y_2^2}-2\frac{\partial^2F_3}{\partial x_2\partial y_2}\mathrm{i}\right)=\frac{3(3k_1^3+17k_1^2+40k_1+60)}{(k_1-6)(k_1-1)^2(k_2+2)^2}+\left(\frac{2\sqrt{-k_1-3}\,(k_1^3+11k_1^2+26k_1+52)}{(k_1-1)^2(k_1+2)^2(k_1-6)}\right)\mathrm{i},$$

$$h_{20}^2=\frac{1}{4}\left(\frac{\partial^2F_4}{\partial x_2^2}-\frac{\partial^2F_4}{\partial y_2^2}-2\frac{\partial^2F_4}{\partial x_2\partial y_2}\mathrm{i}\right)=-\frac{3k_1(3k_1^2+13k_1+8)}{2(k_1-1)^2(k_1+2)^3}+\left(-\frac{k_1\sqrt{-k_1-3}\,(k_1^2+10k_1+7)}{(k_1-1)^2(k_2+2)^3}\right)\mathrm{i}.$$

求解如下方程组:

$$\boldsymbol{D}\boldsymbol{w}_{11}=-\boldsymbol{h}_{11},(\boldsymbol{D}-2\mathrm{i}t_1\boldsymbol{I})\boldsymbol{w}_{20}=-\boldsymbol{h}_{20}. \tag{11}$$

其中:

$$\boldsymbol{D}=\begin{pmatrix}-3&0\\0&-1\end{pmatrix};\boldsymbol{h}_{11}=\begin{pmatrix}h_{11}^1\\h_{11}^2\end{pmatrix};\boldsymbol{h}_{20}=\begin{pmatrix}h_{20}^1\\h_{20}^2\end{pmatrix};\boldsymbol{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

则由方程组(11)可得

$$\boldsymbol{w}_{11}=\begin{pmatrix}\frac{1}{3}h_{11}^1\\h_{11}^2\end{pmatrix};\boldsymbol{w}_{20}=\begin{pmatrix}\frac{-3+2\mathrm{i}\sqrt{-3-k_1}}{3+4k_1}h_{20}^1\\\frac{-1+2\mathrm{i}\sqrt{-3-k_1}}{11+4k_1}h_{20}^2\end{pmatrix}.$$

进一步有

$$G_{110}^1=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_2\partial z_2}+\frac{\partial^2F_2}{\partial y_2\partial z_2}\right)+\left(\frac{\partial^2F_2}{\partial x_2\partial z_2}-\frac{\partial^2F_1}{\partial y_2\partial z_2}\right)\mathrm{i}\right]=\frac{k_1^2+18k_1+48}{-2(k_1+2)(k_1-6)}+\left(\frac{3k_1^2+23k_1+42}{\sqrt{-k_1-3}\,(k+2)(k-6)}\right)\mathrm{i},$$

$$G_{110}^2=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_2\partial w_2}+\frac{\partial^2F_2}{\partial y_2\partial w_2}\right)+\left(\frac{\partial^2F_2}{\partial x_2\partial w_2}-\frac{\partial^2F_1}{\partial y_2\partial w_2}\right)\mathrm{i}\right]=-\frac{k_1^3+14k_1^2+52k_1+24}{k_1(k_1+2)(k_1-6)}+\left(\frac{4k_1^3+42k_1^2+120k_1+72}{k_1\sqrt{-k_1-3}\,(k_1+2)(k_1-6)}\right)\mathrm{i},$$

$$G_{101}^1=\frac{1}{2}\left[\left(\frac{\partial^2F_1}{\partial x_2\partial z_2}-\frac{\partial^2F_2}{\partial y_2\partial z_2}\right)+\left(\frac{\partial^2F_2}{\partial x_2\partial z_2}+\frac{\partial^2F_1}{\partial y_2\partial z_2}\right)\mathrm{i}\right]=\frac{k_1^3+5k_1^2-30k_1-144}{2(k_1-1)(k_1+2)(k_1-6)}+\left(\frac{k_1^3+2k_1^2-45k_1-126}{\sqrt{-k_1-3}\,(k_1-1)(k_1+2)(k_1-6)}\right)\mathrm{i},$$

$$G_{110}^2=\frac{1}{2}\left[\left(\frac{\partial^2 F_1}{\partial x_2\partial w_2}-\frac{\partial^2 F_2}{\partial y_2\partial w_2}\right)+\left(\frac{\partial^2 F_2}{\partial x_2\partial w_2}+\frac{\partial^2 F_1}{\partial y_2\partial w_2}\right)\mathrm{i}\right]=\frac{k_1^3+3k_1^2-16k_1-84}{k_1(k_1-1)(k_1-6)}-\left(\frac{2k_1^2+34k_1+156}{\sqrt{-k_1-3}\,k_1(k_1-1)(k_1-6)}\right)\mathrm{i},$$

$$g_{21}=G_{21}+\sum_{k=1}^{2}(2G_{110}^k w_{11}^k+G_{101}^k w_{20}^k)=\frac{-32k_1^6-172k_1^5+2\,913k_1^4+28\,290k_1^3+75\,162k_1^2+63\,792k_1+16\,632}{(k_1+2)^3(k_1-6)^2(k_1-1)(4k_1+3)(4k_1+11)}-$$

$$\left(\frac{-120k_1^6-338k_1^5+10\,790k_1^4+78\,051k_1^3+188\,400k_1^2+171\,900k_1+52\,272}{\sqrt{-k_1-3}\,(k_1+2)^3(k_1-6)^2(k_1-1)(4k_1+3)(4k_1+11)}\right)\mathrm{i},C_1(0)=\frac{\mathrm{i}}{2w_0}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+$$

$$\frac{1}{2}g_{21}=\frac{-96k_1^7-788k_1^6+8\,095k_1^5+122\,344k_1^4+535\,125k_1^3+986\,334k_1^2+730\,764k_1+182\,952}{6(k_1+2)^3(k_1-6)^2(k_1-1)(k_1+3)(4k_1+11)(4k_1+3)}+$$

$$\frac{1\,064k_1^7+5\,698k_1^6-99\,369k_1^5-1\,087\,886k_1^4-4\,175\,299k_1^3-7\,357\,698k_1^2-5\,750\,676k_1-1\,613\,304}{18\sqrt{-k_1-3}\,(k_1+2)^3(k_1-6)^2(k_1-1)(k_1+3)(4k_1+11)(4k_1+3)}\mathrm{i}.$$

故,$\mu_2=-\dfrac{\mathrm{Re}C_1(0)}{\alpha'(0)}$,$\tau_2=-\dfrac{\mathrm{Im}C_1(0)+\mu_2\omega'(0)}{\omega_0}$,

$\beta_2=2\mathrm{Re}C_1(0)$,$\omega_0=\sqrt{-k_1-3}$,$\alpha'(0)=\mathrm{Re}(\lambda'(k_0))=-\dfrac{1}{2(k_1-6)}$,$\omega_0'(0)=-\dfrac{k_1-3}{2\sqrt{-k_1-3}\,(k_1-6)}$.

系数μ_2,τ_2,β_2具体含义与定理2中(ii)的定义一致.若选取$k_1\in[-\infty,-3.10)$,则$\mu_2>0$且$\beta_2<0$,系统(2)在原点处的Hopf分岔是非退化且超临界的,分岔出的周期解位于$k_2>k_2'$且是稳定的;若选取$k_1\in(-3.09,-3)$,则$\mu_2<0$且$\beta_2>0$,系统(2)在原点处的Hopf分岔是非退化且亚临界的,分岔出的周期解位于$k_2<k_2'$且是不稳定的.

于是,周期解及特征指数为$T=\dfrac{2\pi}{\omega_0}(1+\tau_2\varepsilon^2+O(\varepsilon^4))$,$\beta=\beta_2\varepsilon^2+O(\varepsilon^4)$,$\varepsilon^2=\dfrac{k_2-k_2'}{\mu_2}+O((k_2-k_2')^2)$.

下面证明(iii)成立,事实上

$$\boldsymbol{\xi}=(x,y,z,w)^{\mathrm{T}}=\boldsymbol{P}(y_1,y_2,y_3,y_4)^{\mathrm{T}},y_1=\mathrm{Re}u,y_2=\mathrm{Im}u,(y_3,y_4)^{\mathrm{T}}=\boldsymbol{w}_{11}|u|+\mathrm{Re}(\boldsymbol{w}_{20}u^2)+O(|u|^3),$$

$$u=\varepsilon\mathrm{e}^{\frac{2i\pi}{T}}+\frac{\mathrm{i}\varepsilon^2}{6\omega_0}[g_{02}\mathrm{e}^{-\frac{4i\pi}{T}}-3g_{20}\mathrm{e}^{\frac{4i\pi}{T}}+6g_{11}]+O(\varepsilon^3)=\varepsilon\cos\left(\frac{2\pi t}{T}\right)-\frac{\varepsilon^2(k_1^2+5k_1-12)}{(k_1-1)(k_1+2)(k_1+3)(k_1-6)}-$$

$$\varepsilon^2\left(\frac{\sin\left(\frac{4\pi t}{T}\right)(k_1^4+34k_1^3+139k_1^2+90k_1+96)}{3(k_1-1)^2(k_1+2)^2(k_1-6)\sqrt{-k_1-3}}+\frac{\cos\left(\frac{4\pi t}{T}\right)(9k_1^4+98k_1^3+263k_1^2+182k_1+168)}{3(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right)+$$

$$\left(\varepsilon\sin\left(\frac{2\pi t}{T}\right)+\varepsilon^2\left(\frac{3k_1}{(k_1-1)(k_1+2)(k_1-6)\sqrt{-k_1-3}}\right)\right)\mathrm{i}+$$

$$\varepsilon^2\left(\frac{\cos\left(\frac{4\pi t}{T}\right)(2k_1^4+41k_1^3+125k_1^2-180k_1-348)}{3(k_1-1)(k_1+2)^2(k_1-6)\sqrt{-k_1-3}}-\frac{4\sin\left(\frac{4\pi t}{T}\right)(3k_1^4+28k_1^3+43k_1^2-104k_1-150)}{3(k_1-1)^2(k_1+2)^2(k_1+3)(k_1-6)}\right)\mathrm{i}.$$

经过繁杂计算可得系统(2)在定理2参数条件下的Hopf分岔周期解近似表达式$(x(t),y(t),z(t),w(t))$,如定理2中(iii)所示.

系统(2)取参数$(a,b,c,m,k_1,k_2)=(1,1,0,2,-5,k_2)$时,可得原点为稳定焦点.当$k_1$选取为$-5$,$k_2$选取为0时,$k_2'=-1$.此时,计算得到$\mu_2=0.001\,6$,$\beta_2=-0.036\,26$,且有第一Lyapunov系数$l_1=-0.018\,13$,出现非退化超临界稳定极限环;当选取$k_2=-1.05$及$-0.99$,分别可得稳定焦点及极限环,如图6所示.

3.2　全局动力学的数值分析

选取系统(2)参数$(a,b,c,m,k_2)=(10,8/3,10,1,-2)$,$k_1\in[-250,20]$,则系统(2)呈现超混沌、混沌与周期轨道等多种动力学行为,如表1所示.当$k_1\leqslant-126$或$k_1\in[-55,-45]\cup[-42,-38.5]\cup[-34.5,-13]\cup[-9,-1.5]$时,发生周期或拟周期现象;当$k_1\in[-150,-149.9]\cup[-43,-42.9]\cup[-54.5,-53.5]$时,发生倍周期分岔及反倍周期分岔现象;当$k_1\in[-12.2,-12.1]$时,出现周期窗口;当$k_1\in[0,20]$

时,发生超混沌现象,其分岔如图7所示.选取不同k_1时系统(2)的相图变化如图8所示.

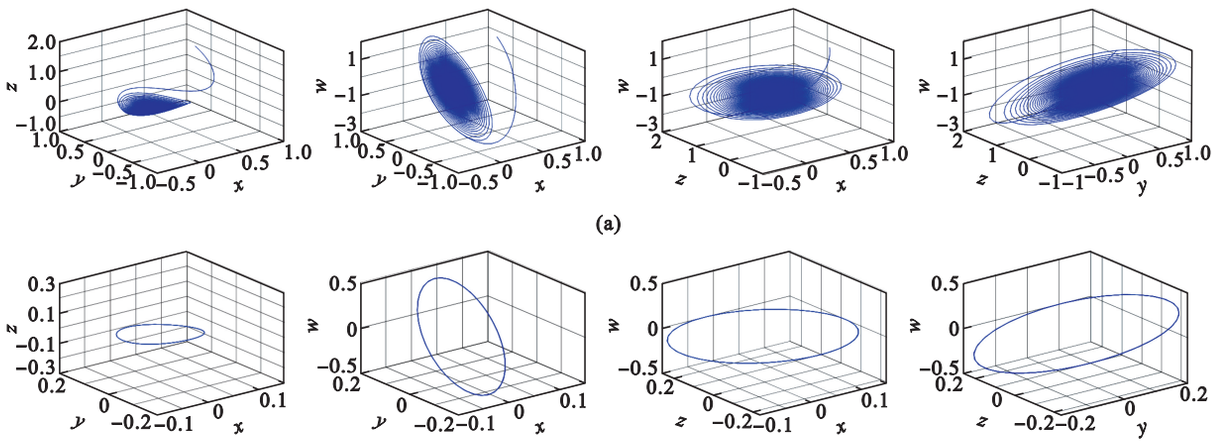


图6 系统(2)原点处Hopf分岔:$(a,b,c,m,k_1)=(1,1,0,2,-5)$

Fig. 6 Hopf bifurcation of system (2) at origin: $(a,b,c,m,k_1)=(1,1,0,2,-5)$

(a)—平衡点,$k_2=-1.05$;(b)—周期解,$k_2=-0.99$.

表1 系统(2)在参数$(a,b,c,m,k_2)=(10,8/3,10,1,-2)$下的动力学行为

Table 1 Dynamical behaviors of system (2) under parameters$(a,b,c,m,k_2)=(10,8/3,10,1,-2)$

k_1	λ_1	λ_2	λ_3	λ_4	Lyapunov维数	动力学行为
−85.2	0.414 0	0.000 0	−3.414 5	−11.640 7	2.121 3	混沌
−62.8	0.000 0	−0.211 3	−2.613 1	−11.798 2	1.000 0	周期窗口
−49.4	0.314 6	0.000 0	−2.266 7	−12.621 1	2.138 8	混沌
13.2	2.604 2	0.116 2	−0.044 7	−17.324 1	3.154 5	超混沌

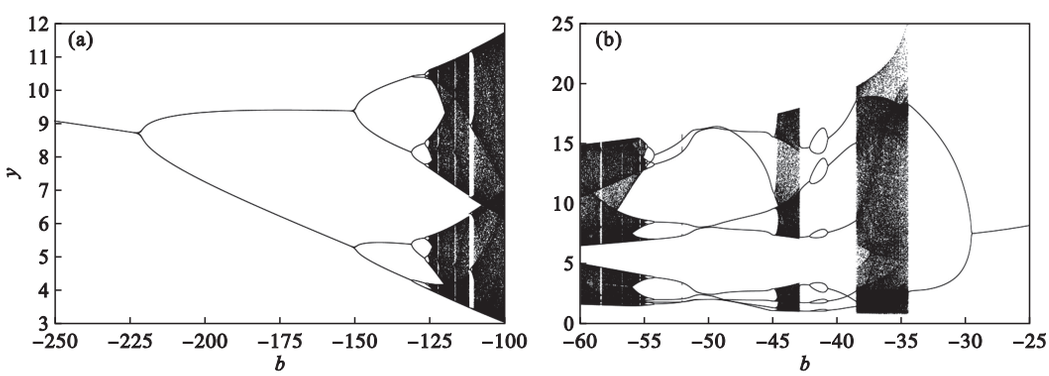
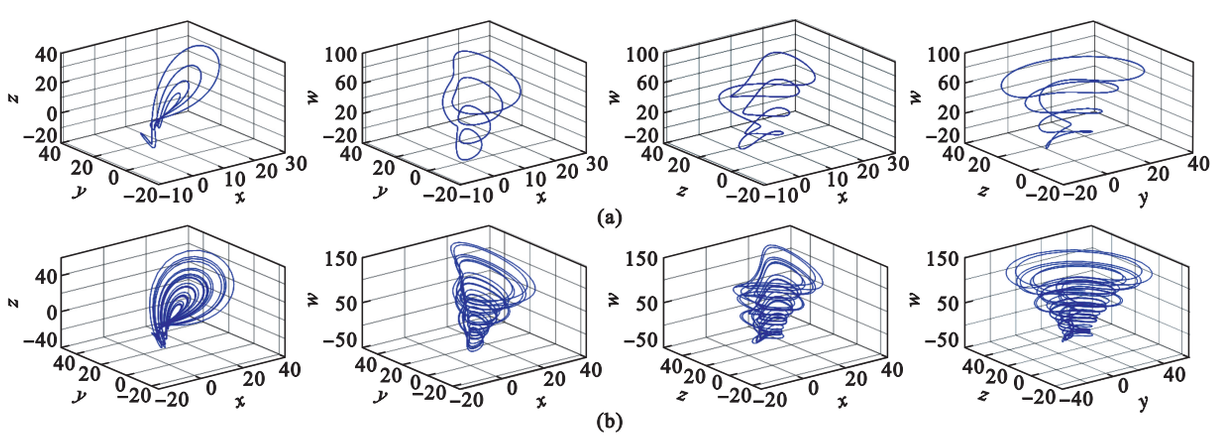
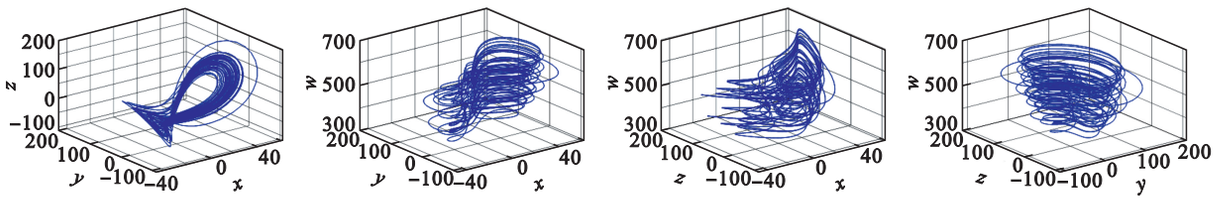


图7 系统(2)的分岔图

Fig. 7 Bifurcation diagrams of system (2)

(a)—$k_1\in[-250,-100]$;(b)—$k_1\in[-60,-25]$.





(c)

图 8　选取不同 k_1 时系统(2)的相图变化

Fig. 8　Phase diagram variations of system (2) under different k_1

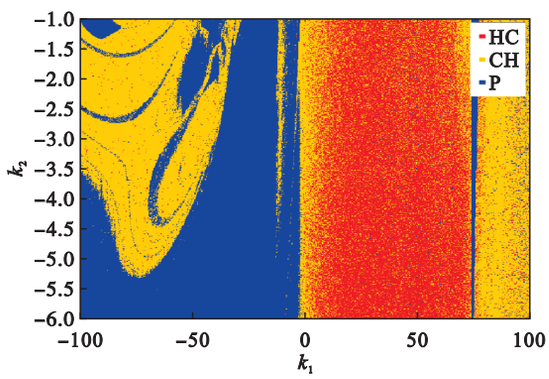
(a)—$k_1=-12.15$; (b)—$k_1=-11$; (c)—$k_1=15$.

若选取参数 $a=10, b=8/3, c=10, m=1, k_1=20, k_2=-2$,初值为(100, 100, 100, 100),变量范围为 $x\in[-300, 700], w\in[-300, 700]$,则系统(2)在 k_1-k_2 平面中发生复杂的动力学行为,如图 9 所示.

图 9 中,平面上存在 3 个不同的区域,分别记为 HC, CH, P. 分隔这些区域的参数值是根据 Lyapunov 指数计算的.HC 区是超混沌区域,用红色表示;CH 区是混沌区域,用黄色表示;P 区是周期轨道区域,用蓝色表示.当 $k_1>0$ 时,系统(2)复杂程度更高,更易发生混沌及超混沌现象.

3.3　电路实现

对于系统(2),本节采用 Multisim 电路软件进行模拟,如图 10 所示设计电路实验验证系统(2)中超混沌与混沌现象的物理存在性.分别取参数 $(a, b, c, m, k_1, k_2)=(15, 5, -25, 1, 15, -19.3)$, $(a, b, c, m, k_1, k_2)=(10, 8/3, 28, 1, -91.6, -2)$ 进行电路实验,得到超混沌和混沌现象电路图像,如图 11 所示.



HC—超混沌; CH—混沌; P—周期轨道.

图 9　系统(2)在 k_1, k_2 参数下的复杂动力学行为

Fig. 9　Complex dynamical phenomena of system (2) under parameters k_1 and k_2

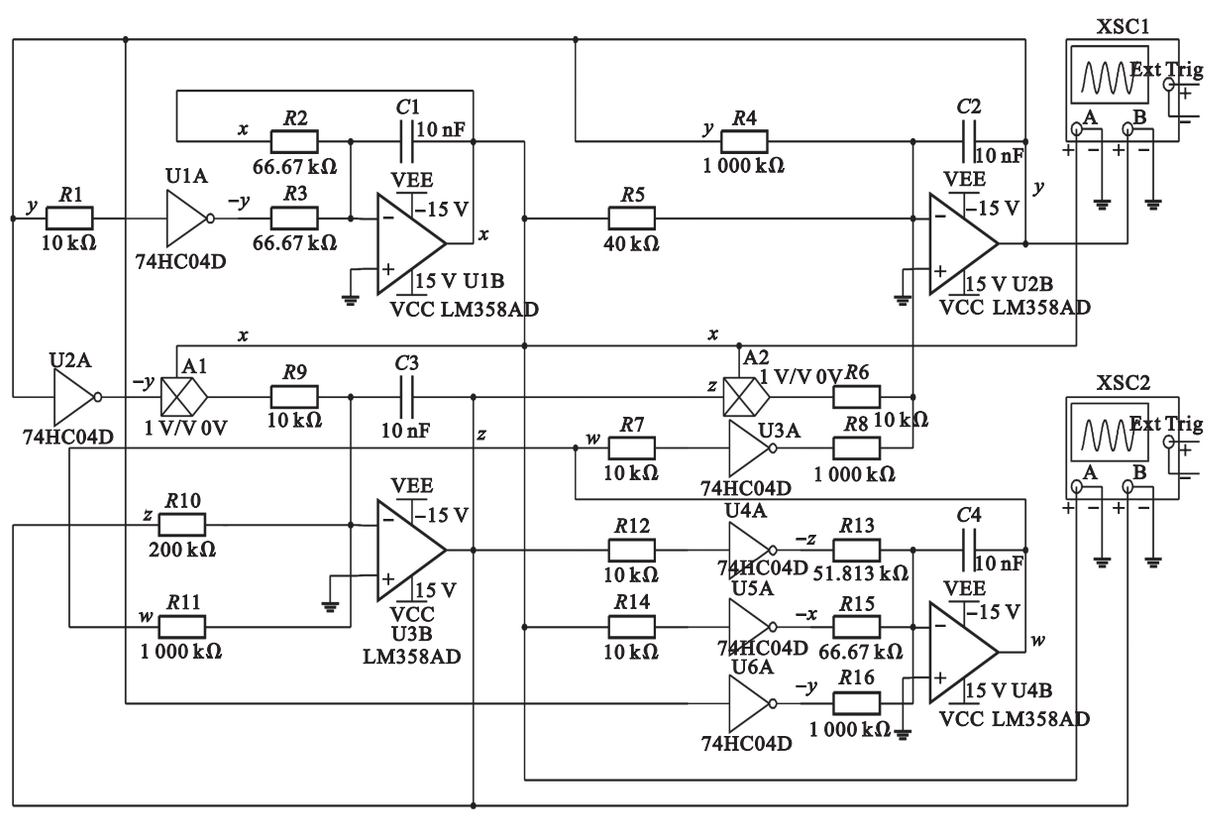


图 10　系统(2)的模拟电路

Fig. 10　Simulated circuit of system (2)

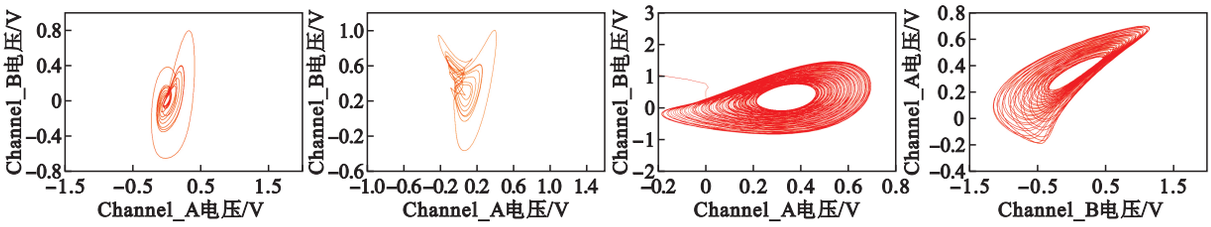


图 11 系统(2)电路实验结果

Fig. 11 Circuit experimental results of system (2)

(a),(b)—混沌截面,参数$(a,b,c,m,k_1,k_2)=(10,8/3,28,1,-91.6,-2)$,初值(1, 1, 1, 1),(a)为$X$-$Y$截面,(b)为$X$-$Z$截面;(c),(d)—超混沌截面,参数$(a,b,c,m,k_1,k_2)=(15,5,-25,1,15,-19.3)$,初值(0, 0, 0, 1 000),(c)为$X$-$Y$截面,(d)为$X$-$Z$截面.

4 结 语

本文基于经典Lorenz系统设计了1个反馈控制器,构建新的四维超混沌系统,数值研究发现该系统具有混沌-环-环面-焦点演变、平衡点曲线、多个吸引子共存等多种有趣的动力学行为.理论分析得到平衡点曲线稳定性的必要条件,讨论平衡点曲线与多吸引子共存间的内在联系,严格证明Hopf分岔的存在性并获得周期解近似表达式.数值分析新的四维超混沌系统参数变化引发的多种分岔类型,并利用电路实验验证超混沌及混沌现象的存在性.尽管如此,该四维超混沌系统仍有许多进一步研究的空间,譬如平衡点曲线的规范化理论亟待建立、多共存吸引子的产生机制仍不清晰、环面与混沌之间的内在机理尚需更好阐明等.同时,该新的四维超混沌系统在图像加密、保密数据传输等工程场景的应用价值尚待后续实验探索.本文将为同类高维超混沌系统的后续分析研究提供一定参考.

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